On the Hedging of Options On Exploding Exchange Rates*

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Abstract

We study a novel pricing operator for complete, local martingale models. The new pricing operator guarantees put-call parity to hold and the value of a forward contract to match the buy-and-hold strategy, even if the underlying follows strict local martingale dynamics. More precisely, we discuss a change of numéraire (change of currency) technique when the underlying is only a local martingale modelling for example an exchange rate. The new pricing operator assigns prices to contingent claims according to the minimal cost for replication strategies that succeed with probability one for both currencies as numéraire. Within this context, we interpret the non-martingality of an exchange-rate as a reflection of the possibility that the numéraire currency may devalue completely against the asset currency (hyperinflation).

1 Introduction

Strict local martingales, that is, local martingales that are not martingales, have recently been introduced in the financial industry to model exchange rates under the risk-neutral measure. This is due to the fact that they are able to capture observed features of the market well such as implied volatiliy surfaces and that they are easily analytically tractable. An important example is the class of "quadratic normal volatility" models, a family of local martingales, which for example are studied in Andersen (2011) and in our companion paper Carr et al. (2012). For a discussion of strict local martingale Foreign Exchange models from a more economic point of view, we refer the reader to Jarrow and Protter (2011).

A standard definition of a contingent claim price in a complete market framework is its minimal superreplicating cost under the risk-neutral measure. As Heston et al. (2007) show, put-call parity then usually does not hold in strict local martingale models. Even more disturbingly, the

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minimal superreplicating price for an asset modelled as a strict local martingale in a complete market is below its current value. However, due to an admissibility constraint on trading strategies, these models do not yield arbitrage opportunities. For example, the strategy of shorting the asset modelled by a strict local martingale and replicating its payoff for a lower cost is not allowed, as it might lead to unbounded negative wealth before the strategy matures.

This implies that pricing contingent claims via minimal superreplicating cost might not be the correct approach for models with asset prices that follow strict local martingale dynamics. Otherwise, put-call parity fails to hold and it is possible to replicate a forward contract by dynamic hedging with lower initial cost than the buy-and-hold replication. However, in real markets put and call prices are usually quoted in implied volatilities yielding directly put-call parity. This mismatch of observed prices in markets and superreplicating prices in arbitrage-free models does not appear to be a paradox but more a lack in the theory of Mathematical Finance.

Cox and Hobson (2005) suggest to consider collateral requirements when pricing contingent claims corresponding to a constraint on the class of admissible trading strategies. This leads to a higher contingent claim price, but usually does not restore put-call parity. So far, only ad-hoc pricing methods have been suggested in order to restore put-call parity. These methods seem to lack a clear economic motivation and thus, cannot easily be extended to general contingent claims. For example, Lewis (2000) proposes to add a correction term to the price of a call. However, as his starting point is exactly put-call parity, it is not clear how other contingent claims should be priced.

Madan and Yor (2006) suggest to take the limit of a sequence of prices obtained from approximating the asset price by true martingales as the price for a contingent claim. This approach also restores put-call parity. However, one might criticize that the limit of the approximating prices does usually not agree with the classical price in the case that the underlying is a true martingale. For instance, a contingent claim that pays infinity in the event that the quadratic variation stays constant for some period and otherwise pays zero has a price of zero if the underlying is geometric Brownian motion, but a price of infinity if the the underlying is stopped Geometric Brownian motion.

In the following, we shall take an economic point of view based on a replicating argument and derive a pricing operator that restores put-call parity and assigns model prices which correspond to the observed market prices. We thus not only justify Lewis' pricing operator by an economic argument but also generalize it to a wider class of models and contingent claims. More precisely, we observe that there exists a natural candidate for a pricing-measure corresponding to a change of numéraire. As we shall discuss below, this is a different approach as the one taken in Delbaen and Schachermayer (1995b) as we do not require the new measure to be equivalent with respect to the original one. If we then define the price of a contingent claim as the minimal superreplicating cost under both measures corresponding to the two different currencies, then these prices satisfy put-parity as we shall demonstrate in Section 3.

This approach can be interpreted as a link between classical pricing and pricing under Knightian uncertainty. Pricing in the classical sense corresponds to the choice of one probability measure under which a contingent claim is superreplicated. This choice implies a strong assumption on the chosen nullsets, that is, by assumption a set of events is determined to be not relevant for computing a replicating trading strategy. If the modeler considered another probability measure, other events would be selected, leading to a different replicating price and strategy. Indeed, one would like that the choice of probability measure should not have a large impact on the price or at least, should be quantifiable.

We compute the minimal cost for replicating a payoff under two probability measures that are not necessarily equivalent with respect to each other. A variety of non-equivalent probability measures is considered in recent research that studies robust trading strategies under *Knightian uncertainty*, which describes a modeler's lack of knowledge of the true probability measure, and in particular, of the nullsets. We refer the reader to Avellaneda et al. (1995), Lyons (1995) and the recent work of Fernholz and Karatzas (2010b) for a discussion of Knightian uncertainty and a literature overview; related is the approach taken in Peng (2010). While these authors study an infinite set of probability measures, we focus on two measures only. Thus, we compromise between the classical theory and the standard Knightian uncertainty studies.

Another interpretation of the pricing operator introduced here is provided in Section 4. Therein, we consider a physical measure under which both currencies might completely devaluate against the other. Clearly, in such a situation an equivalent probability measure cannot exist under which the exchange rate follows local martingale dynamics. However, as one may use both currencies as hedging instruments, superreplication of contingent claims might still be possible. We provide a set of conditions under which replicating strategies exist and we show how the minimal cost for such a strategy is exactly described by the pricing operator of this paper. Towards this end, we introduce a risk-neutral measure which is not equivalent but only absolutely continuous with respect to the physical measure.

This point of view gives us an interpretation of the strict local martingality of an exchange rate under a risk-neutral probability measure as the positive probability of complete devaluations of currencies (corresponding to explosions of the exchange rate) occurring under under some dominating probability measure. We remark that this dominating probability measure does usually not correspond to the Föllmer measure, which we shall discuss below, but is equivalent to the sum of the Föllmer measure and the original measure. Section 4 also contains a review of macro-economic literature discussing hyperinflations, further motivating this point of view.

The mathematical contribution of this paper is mainly contained in Section 2. In order to construct the measure corresponding to a change of numéraire which is allowed to vanish we construct the Föllmer measure for nonnegative local martingales, extending the corresponding results for strictly positive local martingales. We shall provide an overview of the relevant literature in Section 2. We also develop a stochastic calculus for the suggested change of measure, in which neither measure dominates the other one. This will then be used to show that asset price processes denominated in one currency behave "nicely" after a change of currency along with the corresponding change of measure.

2 Change of measure

In this section, we introduce the probabilistic framework of this paper and prove various mathematical tools that we later shall rely on. Most importantly, we present the construction of a probability measure $\widehat{\mathbb{Q}}$ corresponding to a change of numéraire technique. It is helpful to interpret the notation of this section in a financial context. We shall introduce a stochastic process X, which later on will be interpreted as an exchange rate, for example the price of one Euro in Dollars. For sake of simplicity, we shall assume that the interest rates are zero both in the domestic (Dollar) and in the foreign economy (Euro), such that X represents the (Dollar-) price process of a traded asset, namely the European money market account. A probability measure $\mathbb{Q}$ will represent the risk-

neutral measure corresponding to the Dollar-numéraire. In the following sections, the measure $\widehat{\mathbb{Q}}$ then will be interpreted as the risk-neutral measure corresponding to the Euro-numéraire.

2.1 Probabilistic model

Since below we shall study probability measures that are not necessarily equivalent with respect to each other we, a priori, do not specify a probability space but only a measurable space. We start by fixing a finite time horizon $T > 0$, a number of stochastic risk factors $n \in \mathbb{N}$, and the space Ω of all continuous paths $\omega = (\omega^{(1)}, \ldots, \omega^{(n)}) : [0, T] \to [0, \infty] \times \mathbb{R}^{n-1}$, where the first coordinate is nonnegative and gets absorbed in 0 or ∞ whenever it hits one of the two points. Here, and in the following, a function $f : [0, T] \to [0, \infty]$ is called continuous if $\lim_{s \to t, s \in [0,T]} f(s) = f(t)$ for all $t \in [0, T]$.

We define, for each $t \in [0, T]$, the sigma-algebra $\mathcal{F}_t^0 := \bigcap_{s>t} \sigma(\omega_u, u \leq s)$, not completed by the nullsets of some probability measure. This yields the right-continuous modification $\mathbb{F}^0 := (\mathcal{F}_t^0)_{t \in [0,T]}$ of the filtration generated by the paths $\omega \in \Omega$. The filtration $\mathbb{F}^0$ models the flow of information, which is available to all market participants.

To emphasize its importance in the discussion to follow, we denote the first coordinate $\omega^{(1)}$ of ω by $X(\omega)$. By definition, $X = (X_t)_{t \in [0,T]}$ is an $\mathbb{F}^0$-adapted, stochastic process. We also introduce the random times

$$\begin{aligned} R_i &:= \inf\{t \in [0, T] : X_t \geq i\}, \\ S_i &:= \inf \left\{ t \in [0, T] : X_t \leq \frac{1}{i} \right\} \end{aligned} \tag{1}$$

for all $i \in \mathbb{N}$, $R = \lim_{i \uparrow \infty} R_i$, and $S = \lim_{i \uparrow \infty} S_i$, with the convention $\inf \emptyset := T + 1$. All these random times are, despite the incompleteness of $\mathbb{F}^0$, $\mathbb{F}^0$-stopping times; see Problem 1.2.7 and Lemma 1.2.11 of Karatzas and Shreve (1991). We observe that $X_t = 0$ for all $t \geq S$ and $X_t = \infty$ for all $t \geq R$.

Any nonnegative random variable, such as X_T, is explicitly allowed to take values in $[0, \infty]$. For a nonnegative random variable Y and some set $A \in \mathcal{F}_T^0$, we will write $Y \mathbf{1}_A$ to denote the random variable that equals Y whenever $\omega \in A$, and otherwise is zero. To wit, we define $\infty \cdot 0 := 0$.

For any $\mathbb{F}^0$-stopping time τ with values in $[0, T] \cup \{T + 1\}$, we shall denote the stochastic process that arises from stopping a process $N = (N_t)_{t \in [0,T]}$ at time τ by $N^\tau = (N_t^\tau)_{t \in [0,T]}$; that is, $N_t^\tau := N_{t \wedge \tau}$ for all $t \in [0, T]$. We denote the quadratic covariation of two stochastic processes $N^{(1)}$ and $N^{(2)}$, if it exists, by $\langle N^{(1)}, N^{(2)} \rangle = (\langle N^{(1)}, N^{(2)} \rangle)_{t \in [0,T]}$. By convention, we set $\mathcal{F}_{T+1}^0 := \mathcal{F}_T^0$.

We now fix $x_0 > 0$ and assume the existence of a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T^0)$ such that $\mathbb{Q}(R = T + 1) = 1$, $\mathbb{Q}(X_0 = \omega_0^{(1)} = x_0) = 1$, and X is a (nonnegative, continuous) $\mathbb{Q}$-local martingale.

2.2 Stochastic intervals

Throughout this paper, we shall rely on properties of stochastic processes that only hold up to a stopping time, as in the following definition, where $\mathbb{P}$ denotes a generic probability measure:

Definition 1 (Semimartingale / (local) martingale on stochastic interval). For a predictable stopping time τ and a probability measure $\mathbb{P}$ with $\mathbb{P}(\tau > 0) = 1$, we call a stochastic process N a $\mathbb{P}$-semimartingale on $[0, \tau)$ (respectively, on $[0, \tau]$) if for all stopping times $\tilde{\tau}$ with $\tilde{\tau} < \tau$ (respectively, $\tilde{\tau} \leq \tau$) $\mathbb{P}$-almost surely the process $N^{\tilde{\tau}}$ is a $\mathbb{P}$-semimartingale. We shall use the same notation for $\mathbb{P}$-(local) martingales. □

The next lemma characterizes local martingales on a stochastic interval:

Lemma 1 (Localization sequence for a local martingale on a stochastic interval)**.** *Fix a predicable stopping time τ, a probability measure $\mathbb{P}$, and a stochastic process N. Then, N denotes a $\mathbb{P}$-local martingale on $[0, \tau)$ if and only if there exists a sequence of increasing stopping times $(\tau_i)_{i \in \mathbb{N}}$ with $\mathbb{P}(\lim_{i \uparrow \infty} \tau_i = \tau) = 1$ such that N^{τ_i} is a $\mathbb{P}$-martingale. More precisely, for the reverse implication it is sufficient that N^{τ_i} is a $\mathbb{P}$-local martingale only.*

Proof. Let N be a $\mathbb{P}$-local martingale. Since τ is predictable there exists a sequence of increasing stopping times $(\tilde{\tau}_i)_{i \in \mathbb{N}}$ with $\mathbb{P}(\lim_{i \uparrow \infty} \tilde{\tau}_i = \tau) = 1$ and $\tilde{\tau}_i < \tau$ $\mathbb{P}$-almost surely on $\{\tau < T + 1\}$. By assumption, the process $N^{\tilde{\tau}_i} = N^{\tilde{\tau}_i \wedge T}$ is a $\mathbb{P}$-local martingale for all $i \in \mathbb{N}$. Then, a similar argument as in the proof of Theorem I.48(e) in Protter (2003) yields one direction.

For the reverse implication, fix any $\tilde{\tau}$ with $\mathbb{P}(\tilde{\tau} < \tau) = 1$. Since $\mathbb{P}(\lim_{i \uparrow \infty} \tau_i \geq \tilde{\tau}) = 1$ the process $\tilde{N}^{\tilde{\tau}}$ is locally a local martingale, which implies that it is a local martingale; see Theorem I.48(e) of Protter (2003). □

We shall utilize the following technical result, which illustrates that a continuous, nonnegative local martingale on a half-open stochastic interval can be extended to one on a closed interval. For example, the process $N = (N_t)_{t \in [0,T]}$, defined as $N_t := \mathbf{1}_{\{t < \tau\}}$ for all $t \in [0, T]$ and a stopping time τ can be extended to $M = (M_t)_{t \in [0,T]}$ with $M_t := \mathbf{1}_{\{t \leq \tau\}}$, representing a local martingale on $[0, \tau]$.

Proposition 1 (Extension of local martingales on a stochastic interval)**.** *Fix a predicable stopping time τ, a probability measure $\mathbb{P}$, and a continuous, nonnegative $\mathbb{P}$-local martingale N on $[0, \tau)$. Then, there exists a continuous $\mathbb{P}$-local martingale M on $[0, T]$ such that $M = N$ on $[0, \tau)$. The process M is, up to indistinguishability, unique on $[0, \tau]$.*

Proof. The uniqueness of M on $[0, \tau]$ follows directly from its continuity. To show its existence, we observe that the convergence results for nonnegative supermartingales, defined on deterministic intervals, can be extended to stochastic intervals. To see this, we consider a sequence of increasing stopping times $(\tau_i)_{i \in \mathbb{N}}$ as in Lemma 1 with $\mathbb{P}(\tau_i \leq \tau_{i-1} + 1) = 1$ for all $i \in \mathbb{N}$ and $\tau_0 := 0$. We then define a process $\tilde{N} = (\tilde{N}_t)_{t \geq 0}$ by $\tilde{N}_t := N^{\tau_{i+1}}_{\tau_i + t - i}$ for all $t \in [i, i + 1)$ and $t \geq 0$, and equivalently, a corresponding filtration $\tilde{\mathbb{F}}$. The process $\tilde{N}$ is a "stretched out" version of N and easily seen to be a nonnegative martingale. Thus, it converges $\mathbb{P}$-almost surely to some random variable $\tilde{N}_\infty \geq 0$ with $\mathbb{E}^{\mathbb{P}}[\tilde{N}_\infty] < \infty$; see for example Problem 1.3.16 of Karatzas and Shreve (1991). Defining the process $M = (M_t)_{t \in [0,T]}$ by $M_t := N_t$ for all $t \in [0, \tau)$ and $M_t := \tilde{N}_\infty$ for all $t \in [\tau, T]$ thus yields a continuous process.

We now define a sequence of stopping times $\tilde{\tau}_i := \inf\{t \in [0, T] | M_t \geq i\}$ for all $i \in \mathbb{N}$. Due to the continuity of M, there exists some $i^* \in \mathbb{N}$ such that $\tilde{\tau}_i = T + 1$ for all $i \geq i^*$, where i^* might depend on $\omega \in \Omega$. It now is sufficient to check that $M^{\tilde{\tau}_i}$ is a martingale for all $i \in \mathbb{N}$. By definition of M, the process $M^{\tilde{\tau}_i}$ is a martingale on $[0, \tau)$. We again can "stretch out" this process

to the interval $[0, \infty)$ as in the first part of the proof, observe that it is bounded and thus uniformly integrable and therefore can be closed with a last element M^i. However, we have $M^i = M_T$ on the event $\{\tau_i = T + 1\}$. This concludes the proof. $\square$

We refer the reader also to Exercise IV.1.48 in Revuz and Yor (1999), where the case of not necessarily nonnegative local martingales is treated.

2.3 A new measure

Changing the probability measure often reduces the computational effort necessary for the derivation of expected values. For instance, in Mathematical Finance, the price of a contingent claim is often computed through expressing its payoff in a different unit, the so-called ”numéraire,” changing the measure accordingly, and computing the expected payoff under the new measure. However, an essential assumption for the application of Girsanov's theorem, which handles the change of measure, is that the putative density process of the new measure follows martingale dynamics. In this section, we illustrate that it is sufficient if this process is only a nonnegative local martingale, possibly hitting zero. The mathematical tool we rely on is the so-called Föllmer measure, developed in the 1970s for changes of measures with nonnegative supermartingales; see Föllmer (1972).

The Mathematical Finance literature has utilized the Föllmer measure in order to develop a better understanding of strictly positive, local martingales in the context of arbitrage and bubbles; see for example Delbaen and Schachermayer (1995a), Pal and Protter (2010), Fernholz and Karatzas (2010a), Ruf (2012a), and Kardaras et al. (2011). Here, we slightly extend this literature by allowing the local martingale to hit zero. On the other side, true martingales possibly hitting zero as Radon-Nikodym derivatives have been studied by Schönbucher (2000) within the area of Credit Risk. Schönbucher (2000) terms the corresponding measure a "survival measure." We extend this direction of research by allowing the change of measure being determined by a local martingale only.

The next theorem states the main result of this subsection; for the nonnegative $\mathbb{Q}$-local martingale X there exists a probability measure under which X serves as the numéraire. We remark that we only specify the new measure on $(\Omega, \mathcal{F}^0_{R-})$ and not on $(\Omega, \mathcal{F}^0_T)$. This is due to the fact that the original measure $\mathbb{Q}$, by assumption, does not "see" any events after the stopping time R. Thus, a measure on $(\Omega, \mathcal{F}^0_T)$ satisfying the properties of the next theorem could be easily constructed by arbitrarily, but consistently, extending the measure $\widehat{\mathbb{Q}}$ from $\mathcal{F}^0_{R-}$ to $\mathcal{F}^0_T$.

Theorem 1 (Existence of numéraire measure)**.** *There exists a probability measure $\widehat{\mathbb{Q}}$ on $(\Omega, \mathcal{F}^0_T)$, unique on $(\Omega, \mathcal{F}^0_{R-})$, with corresponding expectation operator $\mathbb{E}^{\widehat{\mathbb{Q}}}$, such that for any $\mathbb{F}^0$-stopping time τ*

$$\widehat{\mathbb{Q}}\left(A \bigcap \left\{\frac{1}{X^\tau_T} > 0\right\}\right) = \frac{\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_A X^\tau_T\right]}{x_0} \tag{2}$$

for any $A \in \mathcal{F}^0_\tau$; thus

$$\mathbb{E}^{\widehat{\mathbb{Q}}}\left[Y \mathbf{1}_{\{1/X^\tau_T > 0\}}\right] = \frac{\mathbb{E}^{\mathbb{Q}}\left[(Y \mathbf{1}_{\{X^\tau_T > 0\}}) X^\tau_T\right]}{x_0} \tag{3}$$

for any $\mathcal{F}^0_\tau$-measurable random variable $Y \geq 0$.

Moreover, $\widehat{\mathbb{Q}}(S = T+1) = 1$, the process $1/X$ is a $\widehat{\mathbb{Q}}$-local martingale, and $\mathbb{Q}$ is absolutely continuous with respect to $\widehat{\mathbb{Q}}$ on $\mathcal{F}^0_{S_i}$ with

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\widehat{\mathbb{Q}}}\right|_{\mathcal{F}^0_{S_i}} = \frac{x_0}{X_T^{S_i}} \tag{4}$$

for all $i \in \mathbb{N}$.

Proof. Föllmer (1972) in conjunction with Meyer (1972) yields the existence and uniqueness of a probability measure $\widehat{\mathbb{Q}}$ on $(\Omega, \mathcal{F}^0_{R-})$ such that (2) holds; see page 123 of Meyer (1972) and Lemma 1.8 and Proposition 2.1 of Föllmer (1972). More precisely, for the existence of this probability measure, we first consider the space $\widetilde{\Omega}$ of all continuous paths $\omega = (\omega^{(1)}, \ldots, \omega^{(n)}) : [0,T] \rightarrow [0,\infty] \times [-\infty,\infty]^{n-1}$, which stay constant after either the first component hits either zero or infinity or one of the last $n-1$ components hits either negative or positive infinity. Similar to above, we define the sigma-algebra $\widetilde{CF}^0_t := \bigcap_{s>t} \sigma(\omega_u, u \leq s)$. This yields a standard system $(\widetilde{\Omega}, \{\widetilde{CF}^0_t\}_{t\in[0,T]})$, for which the construction by Föllmer (1972) can be applied to. Having constructed the Föllmer measure, we observe that we can now embed the measurable space to the original space $(\Omega, \mathbb{F}^0)$, yielding the first part of the statement.

Proposition 4.2 in Föllmer (1972) implies that $1/X$ is a (nonnegative) $\widehat{\mathbb{Q}}$-supermartingale, which in particular implies $\widehat{\mathbb{Q}}(S = T+1) = 1$. Now, fix $i \in \mathbb{N}$, $t \in [0,T]$, $s \in [0,t]$, and $A \in \mathcal{F}^0_{s\wedge S_i}$. Then, (3) applied with $\tau = S_i \wedge t$ and with $\tau = S_i \wedge s$ yields

$$\begin{aligned}
\mathbb{E}^{\widehat{\mathbb{Q}}}\left[\frac{x_0}{X_t^{S_i}}\mathbf{1}_A\right] &= \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\frac{x_0}{X_T^{S_i\wedge t}}\mathbf{1}_A\mathbf{1}_{\{1/X_T^{S_i\wedge t}>0\}}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_A\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{X_T^{S_i\wedge s}}\mathbf{1}_A X_T^{S_i\wedge s}\right] \\
&= \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\frac{x_0}{X_s^{S_i}}\mathbf{1}_A\right],
\end{aligned} \tag{5}$$

which shows that $1/X^{S_i}$ is a $\widehat{\mathbb{Q}}$-martingale. Thus, $1/X$ is a $\widehat{\mathbb{Q}}$-local martingale. Reading (5) with $t = T$ and $A \in \mathcal{F}^0_{S_i}$ yields (4). □

As a technical detail, we remark that, despite the continuous sample paths, $\mathcal{F}_{R-}$ usually is a proper subset of $\mathcal{F}_R$, as we work with the right-continuous modification of the filtration. Thus, we do not expect to have uniqueness of $\widetilde{\mathbb{Q}}$ on $\mathcal{F}_R$. This technical issue, however, shall not be relevant in the further discussion as there will be a natural extension of $\widetilde{\mathbb{Q}}$ to $\mathcal{F}_R$ and to $\mathcal{F}_T$ as we discuss below.

It is important to note that the two measures $\mathbb{Q}$ and $\widehat{\mathbb{Q}}$ are usually not absolutely continuous with respect to each other. This is further discussed in Remark 1 below. In particular, we in general have that $\widehat{\mathbb{Q}}(R = T+1) < 1$. Also, the stopping times S_i in (4) can usually not be replaced by T. We furthermore note that the indicator of the event $\{X_T^\tau > 0\}$ can be omitted in (3) if either Y

is finite or X strictly positive $\mathbb{Q}$-almost surely. In general, however, it is necessary as the example $Y = 1/X_T$ illustrates.

We now discuss how the martingality of X under $\mathbb{Q}$ and $1/X$ under $\widehat{\mathbb{Q}}$ is related to the absolute continuity of the two measures with respect to each other:

Remark 1 (Duality of strict local martingality and density processes hitting zero). If X is not only a $\mathbb{Q}$-local martingale but a true martingale, then the standard Girsanov theorem (see Theorem VIII.1.4 of Revuz and Yor, 1999) yields that $\widehat{\mathbb{Q}}$ is absolutely continuous with respect to $\mathbb{Q}$ on the space $(\Omega, \mathcal{F}_T^0)$ with Radon-Nikodym derivative X_T/x_0. Furthermore, using $A = \Omega$ and $\tau = T$ in (2) yields directly $\widehat{\mathbb{Q}}(1/X_T > 0) = 1$. The same argument shows that $\widehat{\mathbb{Q}}(1/X_T > 0) < 1$ if X is a strict $\mathbb{Q}$-local martingale, in which case $\widehat{\mathbb{Q}}$ is also not absolutely continuous with respect to $\mathbb{Q}$. In particular, $\widehat{\mathbb{Q}}(R = T+1) = 1$ if and only if X is a $\mathbb{Q}$-martingale.

The other direction also holds true; $1/X$ is a true $\widehat{\mathbb{Q}}$-martingale and $\mathbb{Q}$ is absolutely continuous with respect to $\widehat{\mathbb{Q}}$ on $(\Omega, \mathcal{F}_T^0)$ if and only if $\mathbb{Q}(X_T > 0) = 1$. This can be seen directly from (3) with $Y = 1/X_T \mathbf{1}_{\{X_T>0\}}$ and $\tau = T$. $\square$

In the following, we shall derive several properties of the measure change in Theorem 1. In particular, we shall focus on understanding which of the martingale properties of stochastic processes survive the change of measure (possibly after modifying the processes) and how semimartingale dynamics depend on the choice of measure.

In order to describe conditional changes of measures, we formulate a Bayes' formula. If X is a $\mathbb{Q}$-martingale, this has been well-known; see for example Lemma 3.5.3 in Karatzas and Shreve (1991). If X is a strictly positive, $\mathbb{Q}$-local martingale, the Bayes' formula has been derived in Ruf (2012a).

Proposition 2 (Bayes' formula)**.** *For all stopping times ρ, τ with $\rho \leq \tau \leq T$ $\mathbb{Q}$- and $\widehat{\mathbb{Q}}$-almost surely and for all $\mathcal{F}_\tau^0$-measurable random variables $Y \geq 0$ we have the Bayes' formula*

$$\mathbb{E}^{\widehat{\mathbb{Q}}}\left[\left.\frac{Y\mathbf{1}_{\{1/X_\tau>0\}}}{X_\tau}\right| \mathcal{F}_\rho^0\right]\mathbf{1}_{\{X_\rho>0\}} = \mathbb{E}^{\mathbb{Q}}\left[Y\mathbf{1}_{\{X_\tau>0\}}\middle| \mathcal{F}_\rho^0\right]\mathbf{1}_{\{1/X_\rho>0\}}\frac{1}{X_\rho} \tag{6}$$
$$\left(= \mathbb{E}^{\mathbb{Q}}\left[Y\mathbf{1}_{\{X_\tau>0\}}\middle| \mathcal{F}_\rho^0\right]\mathbf{1}_{\{1/X_\rho>0\}}\frac{1}{X_\rho}\mathbf{1}_{\{X_\rho>0\}}\right).$$

This equality holds $\mathbb{Q}$- and $\widehat{\mathbb{Q}}$-almost surely.

Proof. WLOG, we assume $x_0 = 1$. We fix an $A \in \mathcal{F}_\rho^0$. Then, we obtain from $\widehat{\mathbb{Q}}(X_\rho > 0) = 1$ and

$$\begin{aligned}\mathbb{E}^{\widehat{\mathbb{Q}}}\left[\mathbf{1}_A\frac{Y\mathbf{1}_{\{1/X_\tau>0\}}}{X_\tau}\right] &= \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_A Y\mathbf{1}_{\{X_\tau>0\}}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_A\mathbb{E}^{\mathbb{Q}}[Y\mathbf{1}_{\{X_\tau>0\}}|\mathcal{F}_\rho^0]\right] \\ &= \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\mathbf{1}_A\mathbb{E}^{\mathbb{Q}}[Y\mathbf{1}_{\{X_\tau>0\}}|\mathcal{F}_\rho^0]\mathbf{1}_{\{1/X_\rho>0\}}\frac{1}{X_\rho}\right]\end{aligned}$$

that (6) holds $\widehat{\mathbb{Q}}$-almost surely. Now, we obtain from observing that $\mathbb{Q}(1/X_\rho > 0) = 1$ and

$$\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_A\frac{Y\mathbf{1}_{\{X_\tau>0\}}}{X_\rho}\right] = \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\mathbf{1}_A\frac{Y}{X_\rho X_\tau}\mathbf{1}_{\{1/X_\tau>0\}}\right]$$

$$= \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\mathbf{1}_A \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\left.\frac{Y\mathbf{1}_{\{1/X_\tau>0\}}}{X_\tau}\right|\mathcal{F}^0_\rho\right]\frac{1}{X_\rho}\right]$$
$$= \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_A \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\left.\frac{Y\mathbf{1}_{\{1/X_\tau>0\}}}{X_\tau}\right|\mathcal{F}^0_\rho\right]\mathbf{1}_{\{X_\rho>0\}}\right].$$

that (6) holds $\mathbb{Q}$-almost surely. ☐

In the discussion to follow, we shall rely on the next lemma:

Lemma 2 (Convergence of stopping times)**.** *For any sequence of increasing stopping times* $(\tau_i)_{i\in\mathbb{N}}$ *we have that* $\mathbb{Q}(\lim_{i\uparrow\infty}\tau_i \geq S) = 1$ *if and only if* $\widehat{\mathbb{Q}}(\lim_{i\uparrow\infty}\tau_i \geq R) = 1$.

Proof. Fix any sequence of increasing stopping times $(\tau_i)_{i\in\mathbb{N}}$ with $\mathbb{Q}(\lim_{i\uparrow\infty}\tau_i \geq S) = 1$. If $\widehat{\mathbb{Q}}(\lim_{i\uparrow\infty}\tau_i < R) > 0$ then there exists some $j\in\mathbb{N}$ such that $\widehat{\mathbb{Q}}(A_j) > 0$ for $A_j := \{\lim_{i\uparrow\infty}\tau_i < R_j \wedge S_j\}$. However, with $\tau = R_j$, this contradicts (2) since $\mathbb{Q}(A_j) = 0$. The other direction follows equivalently. ☐

We need the Bayes' formula of Proposition 2 in particular for the next set of equivalences and implications. We note that the statements in Remark 1 also can be obtained from using $\tau = T$, $N = 1$ and $N = X$ in (i) of the following observation:

Proposition 3 (Equivalence of (local) martingales)**.** *Let* $N = (N_t)_{t\in[0,T]}$ *be a nonnegative, progressively measurable process and* τ *a stopping time. We then have*

(i) $N^\tau\mathbf{1}_{\{X^\tau>0\}}$ *is a* $\mathbb{Q}$*-martingale if and only if* $(N^\tau\mathbf{1}_{\{1/X^\tau>0\}})/X^\tau$ *is a* $\widehat{\mathbb{Q}}$*-martingale;*

(ii) N *is a* $\mathbb{Q}$*-local martingale on* $[0,S)$ *if and only if* N/X *is a* $\widehat{\mathbb{Q}}$*-local martingale on* $[0,R)$.

If additionally $N(\omega)$ *is continuous for all* $\omega\in\Omega$ *then we also have*

(iii) if N *is a* $\mathbb{Q}$*-martingale on* $[0,S)$*, then* N/X *is a* $\widehat{\mathbb{Q}}$*-local martingale with* $N_R/X_R = 0$ *on* $\{R\leq T\}$*,*

(iv) if N *is a* $\widehat{\mathbb{Q}}$*-martingale on* $[0,R)$*, then* NX *is a* $\mathbb{Q}$*-local martingale with* $N_SX_S = 0$ *on* $\{S\leq T\}$*.*

Proof. The statement in (i) is a corollary of Proposition 2 if we replace τ by $\tau\wedge t$ and use $Y = N^\tau_t$ and $\rho = \tau\wedge s$ in (6) for all $t\in[0,T]$ and $s\in[0,t]$.

Assume now that N is a $\mathbb{Q}$-local martingale on $[0,S)$. By Lemma 1 there exists a sequence of increasing stopping times $(\tau_i)_{i\in\mathbb{N}}$ with $\mathbb{Q}(\lim_{i\uparrow\infty}\tau_i = S) = 1$ such that N^{τ_i} is a $\mathbb{Q}$-martingale. Now, set $\tilde{\tau}_i := \tau_i\wedge S_i$ for all $i\in\mathbb{N}$. Then, (i) implies that $(N^{\tilde{\tau}_i}\mathbf{1}_{\{1/X^{\tilde{\tau}_i}>0\}})/X^{\tilde{\tau}_i}$ is a $\widehat{\mathbb{Q}}$-martingale. Now, another application of Lemma 1 in conjunction with Lemma 2 yields that $(N\mathbf{1}_{\{1/X>0\}})/X$ is a $\widehat{\mathbb{Q}}$-local martingale on $[0,R)$. This shows that N/X is a $\widehat{\mathbb{Q}}$-local martingale on $[0,R)$. The other direction follows in the same manner.

For the statement in (iii), we first use (ii) to see that $\widetilde{N} := N/X$ is a $\widehat{\mathbb{Q}}$-local martingale on $[0,R)$, and thus, due to its continuity and the uniqueness of its extension M in Proposition 1, also on $[0,R]$. Assume now that $\widetilde{N}_R > 0$ on some $A\subset\{R\leq T\}$ with $\widehat{\mathbb{Q}}(A) > 0$. Since we have $\widehat{\mathbb{Q}}(\lim_{i\uparrow\infty}S_i = S = T+1) = 1$ we can assume, without loss of generality, $A\subset\{R\leq T\wedge S_i\}$

for some $i \in \mathbb{N}$. Now, applying (i) yields that $\widetilde{M} := (N^{S_i}\mathbf{1}_{\{1/X^{S_i}>0\}})/X^{S_i} = \widetilde{N}^{S_i}\mathbf{1}_{\{1/X^{S_i}>0\}}$ is a $\widehat{\mathbb{Q}}$-martingale. We observe $\widehat{\mathbb{Q}}(\widetilde{M}_R^{S_i} \leq \widetilde{N}_R^{S_i}) = 1$ and $\widehat{\mathbb{Q}}(\widetilde{M}_R^{S_i} < \widetilde{N}_R^{S_i}) \geq \widehat{\mathbb{Q}}(A) > 0$, contradicting the martingale property of $\widetilde{M}$ under $\widehat{\mathbb{Q}}$. In the same manner, (iv) can be proven. $\square$

By considering the trivial case $X \equiv 1$, it is clear that the reverse directions in (iii) and (iv) of the last proposition cannot hold in general. We also observe that the statement in (i) can be rewritten as follows:

(i') $N^\tau \mathbf{1}_{\{X^\tau>0\}}$ is a $\mathbb{Q}$-martingale on $[0,S]$ if and only if $(N^\tau \mathbf{1}_{\{1/X^\tau>0\}})/X^\tau$ is a $\widehat{\mathbb{Q}}$-martingale on $[0,R]$.

If X is a true $\mathbb{Q}$-martingale, then $\widehat{\mathbb{Q}}(R = T+1) = 1$ as discussed in Remark 1, and the statements of Proposition 3 are just the standard Girsanov-type results, as derived in Van Schuppen and Wong (1974).

The next statement utilizes on the observations that the two measures $\mathbb{Q}$ and $\widehat{\mathbb{Q}}$ are locally equivalent. Thus, one can locally apply Girsanov's theorem to derive the dynamics of several processes under $\widehat{\mathbb{Q}}$.

Proposition 4 (Girsanov-type properties)**.** *Any continuous $\mathbb{Q}$-semimartingale on $[0,S)$ is a continuous $\widehat{\mathbb{Q}}$-semimartingale on $[0,R)$. More precisely,*

(i) for any given continuous $\mathbb{Q}$-local martingale N on $[0,S)$, the process $\widetilde{N} = (\widetilde{N}_t)_{t\in[0,T]}$ defined as

$$\widetilde{N}_t := \left(N_t - \int_0^t \frac{1}{X_s}\mathrm{d}\langle N, X\rangle_s\right)\mathbf{1}_{\{t<R\wedge S\}} \tag{7}$$

for all $t \in [0,T]$ is a $\widehat{\mathbb{Q}}$-local martingale on $[0,R)$; in particular, N/X is a $\widehat{\mathbb{Q}}$-local martingale on $[0,R)$ with dynamics

$$\mathrm{d}\frac{N_t}{X_t} = \frac{1}{X_t}\mathrm{d}\widetilde{N}_t - \frac{N_t}{X_t^2}\mathrm{d}\widetilde{X}_t \tag{8}$$

for all $t \in [0,R)$ where $\widetilde{X}$ is defined as in (7)*;*

(ii) if $N^{(1)}$ and $N^{(2)}$ are two continuous $\mathbb{Q}$-local martingales on $[0,S)$, then $\langle N^{(1)}, N^{(2)}\rangle = \langle \widetilde{N}^{(1)}, \widetilde{N}^{(2)}\rangle$ on $[0,R\wedge S)$ $\mathbb{Q}$- and $\widehat{\mathbb{Q}}$-almost surely, where $\widetilde{N}^{(1)}$ and $\widetilde{N}^{(2)}$ are defined as in (7).

Proof. We consider a $\mathbb{Q}$-local martingale N on $[0,S)$ with a localization sequence $(\tau_i)_{i\in\mathbb{N}}$ as described in Lemma 1. By Lemma 2 the stopping times $\tilde{\tau}_i := \tau_i \wedge R_i$ satisfy $\widehat{\mathbb{Q}}(\lim_{i\uparrow\infty}\tilde{\tau}_i = R) = 1$. We then observe that $N^{\tilde{\tau}_i}$ is $\mathcal{F}^0_{R_i}$-measurable and by (2), $\widehat{\mathbb{Q}}$ is absolutely continuous with respect to $\mathbb{Q}$ on $\mathcal{F}^0_{R_i}$. Thus, by the standard Girsanov's theorem, applied on the measure space $(\Omega, \mathcal{F}^0_{R_i})$, $\widetilde{N}^{\tilde{\tau}_i}$ is a local martingale for all $i \in \mathbb{N}$; see Theorem VIII.1.4 of Revuz and Yor (1999). Then, another application of Lemma 1 shows the first part of (i).

For the second part of (i), we have that N/X is a $\widehat{\mathbb{Q}}$-semimartingale on $[0,R)$. An application of Itô's formula then shows (8), which proves that it is a $\widehat{\mathbb{Q}}$-local martingale on $[0,R)$.

Finally, (ii) follows since adding a finite variation component to a stochastic processes does not change its quadratic covariation with another process. $\square$

In order better to understand the suggested change of measure here, it is instructive to study an extreme case where the measures $\mathbb{Q}$ and $\widehat{\mathbb{Q}}$ are not only not absolutely continuous with respect to each other but even singular:

Example 1 (Singular measures I). Let X denote a nonnegative $\mathbb{Q}$-local martingale with quadratic variation $\langle X\rangle_t = 1/(T-t)\mathbf{1}_{\{t\leq S\}}$ and $X_0 = 1$. Then, there exists a Brownian motion $W = (W_t)_{t\in[0,T]}$ (stopped at time S) such that

$$X_t = 1 + \int_0^{t\wedge S} \frac{1}{\sqrt{T-u}}\mathrm{d}W_u \tag{9}$$

for all $t \in [0,T)$; see the arguments in Section 4.5 of Stroock and Varadhan (2006). Since X corresponds to a deterministically time-changed Brownian motion, we have $\mathbb{Q}(S < T) = 1$ and thus $\mathbb{Q}(X_T = 0) = 1$. Now, under the measure $\widehat{\mathbb{Q}}$ of Theorem 1, we compute that $Y := 1/X$ has the dynamics

$$\mathrm{d}Y_t = -Y_t^2 \frac{1}{\sqrt{T-t}}\mathrm{d}W_t^{\widehat{\mathbb{Q}}} \tag{10}$$

for all $t \in [0,R)$ and some $\widehat{\mathbb{Q}}$-Brownian motion $W^{\widehat{\mathbb{Q}}} := (W_t^{\widehat{\mathbb{Q}}})_{(t\in[0,R))}$; see (i) of Proposition 4. Thus, under $\widehat{\mathbb{Q}}$, the process Y is just the time-change of the reciprocal of a three-dimensional Bessel process Z starting in one. More precisely, define processes $Z = (Z_u)_{u\geq 0}$ and $B = (B_u)_{u\geq 0}$ by $Z_u := Y_{T(1-\exp(-u))}$ and

$$B_u := \int_0^{T(1-\exp(-u))\wedge S} \frac{1}{\sqrt{T-v}}\mathrm{d}W_v^{\widehat{\mathbb{Q}}}$$

and observe that $\mathrm{d}Z_u = -Z_u^2\mathrm{d}B_u$ and $\langle B\rangle_u = u$ for all $u \geq 0$ by Lévy's theorem; see Theorem 3.3.16 in Karatzas and Shreve (1991). We then obtain $Y_t = Z_{\log(T/(T-t))}$, $Y_t > 0$ for all $t \in [0,T)$ and $Y_T = 0$; thus, $\widehat{\mathbb{Q}}(R = T) = 1$; see Section 3.3 of Karatzas and Shreve (1991).

Indeed, the two measures are singular with respect to each other on $\mathcal{F}^0(T)$ since $\mathbb{Q}(R = T) = 0 < 1 = \widehat{\mathbb{Q}}(R = T)$; however, $\widehat{\mathbb{Q}}$ is absolutely continuous with respect to $\mathbb{Q}$ on $\mathcal{F}^0(t)$ for all $t \in [0,T)$ since Y is a strictly positive, strict $\widehat{\mathbb{Q}}$-local martingale; see Remark 1. We also note that X_t is a true $\mathbb{Q}$-martingale on $[0,t]$ for all $t \in [0,T)$, but a strict $\mathbb{Q}$-local martingale on $[0,T]$. $\square$

The next example is a slight modification of the example in Delbaen and Schachermayer (1998). It illustrates here that the equivalence of two probability measures $\mathbb{Q}^{(1)}$ and $\mathbb{Q}^{(2)}$ on $(\Omega, \mathcal{F}_T^0)$ does not necessarily imply the equivalence of the corresponding probability measures $\widehat{\mathbb{Q}}^{(1)}$ and $\widehat{\mathbb{Q}}^{(2)}$ constructed in Theorem 1. This observation will be one reason why we shall assume complete markets later on.

Example 2 (Lack of equivalence). On any probability space, denote two independent processes with the same distribution as the process in (9) by $Z^{(i)} := (Z_t^{(i)})_{t\in[0,T]}$ and define the stopping times

$$\tau^{(i)} := \inf\left\{t \in [0,T] : Z_t^{(i)} \leq \frac{1}{2}\right\}$$

for $i = 1, 2$. On the canonical path space $(\Omega, \mathcal{F}_T^0)$ of Subsection 2.1 for $n = 2$, we define the probability measure $\mathbb{Q}^{(1)}$ as the one induced by the paths of $(Z^{(1),\tau^{(1)}\wedge\tau^{(2)}}, Z^{(2),\tau^{(1)}\wedge\tau^{(2)}})$. Since $\tau^{(2)} < T$ $\mathbb{Q}^{(1)}$-almost surely, the process $X = \omega^{(1)}$ is a strictly positive true $\mathbb{Q}^{(1)}$-martingale and so is $\omega^{(2)}$.

Now, we define a new probability measure $\mathbb{Q}^{(2)}$ by $\mathrm{d}\mathbb{Q}^{(2)}/\mathrm{d}\mathbb{Q}^{(1)} = \omega_T^{(2)}$. We observe that the process $1/\omega^{(2)}$ is a bounded $\mathbb{Q}^{(2)}$-martingale stopped at the minimum of the two stopping times

$$\tilde{\tau}^{(1)} := \inf\left\{t \in [0, T] : X_t \leq \frac{1}{2}\right\},$$
$$\tilde{\tau}^{(2)} := \inf\left\{t \in [0, T] : 1/w_t^{(2)} \geq 2\right\};$$

the process X is only a strict $\mathbb{Q}^{(2)}$-local martingale since

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^{(2)}}[X_T] &= \frac{1}{2}\mathbb{Q}^{(2)}(\tilde{\tau}^{(1)} < \tilde{\tau}^{(2)}) + \mathbb{E}^{\mathbb{Q}^{(1)}}\left[X_T \mathbf{1}_{\{\tilde{\tau}^{(1)} > \tilde{\tau}^{(2)}\}} w_T^{(2)}\right] \\
&= \frac{1}{2}\mathbb{E}^{\mathbb{Q}^{(1)}}\left[\mathbf{1}_{\{\tilde{\tau}^{(1)} < \tilde{\tau}^{(2)}\}} w_T^{(2)}\right] + \mathbb{E}^{\mathbb{Q}^{(1)}}\left[X_T \mathbf{1}_{\{\tilde{\tau}^{(1)} < \tilde{\tau}^{(2)}\}} w_T^{(2)}\right] \\
&< \frac{1}{2} + \mathbb{E}^{\mathbb{Q}^{(2)}}\left[X_T \mathbf{1}_{\{\tilde{\tau}^{(1)} < \tilde{\tau}^{(2)}\}}\right] \\
&< 1.
\end{aligned}$$

However, $\mathbb{Q}^{(1)}$ and $\mathbb{Q}^{(2)}$ are equivalent since $\omega^{(2)}$ is $\mathbb{Q}^{(1)}$- and $\mathbb{Q}^{(2)}$-almost surely strictly positive.

Let $\widehat{\mathbb{Q}}^{(1)}$ and $\widehat{\mathbb{Q}}^{(2)}$ now denote the probability measures of Theorem 1 with $\mathrm{d}\mathbb{Q}^{(i)}/\mathrm{d}\widehat{\mathbb{Q}}^{(i)} = 1/X_T$ for $i = 1, 2$. It is clear that these two measures cannot be equivalent since X is a strictly positive true $\mathbb{Q}^{(1)}$-martingale, and thus, $\mathbb{Q}^{(1)}$ is equivalent to $\widehat{\mathbb{Q}}^{(1)}$; however, $\mathbb{Q}^{(2)}$ and $\widehat{\mathbb{Q}}^{(2)}$ are not equivalent since X is only a strict $\mathbb{Q}^{(2)}$-local martingale.

To elaborate on this, under both measures $\widehat{\mathbb{Q}}^{(1)}$ and $\widehat{\mathbb{Q}}^{(2)}$, the process $1/X$ is a martingale and follows the same dynamics as the process Y in (10), stopped in $\tilde{\tau}^{(1)} \wedge \tilde{\tau}^{(2)}$, which is the first time that either $1/X$ hits 2 or $\omega^{(2)}$ hits $1/2$. However, the distribution of $\tilde{\tau}^{(2)}$ varies under the two measures. Loosely speaking, under $\widehat{\mathbb{Q}}^{(1)}$, the stopping time $\tilde{\tau}^{(2)}$ is the first time that a nonnegative local martingale starting in 1 with the dynamics of (9) hits $1/2$ which happens almost surely before time T; under $\widehat{\mathbb{Q}}^{(2)}$, it is the first time that a nonnegative martingale starting in 1 hits 2 which does not happen almost surely. Thus, we obtain $\widehat{\mathbb{Q}}^{(1)}(1/X_T = 0) = 0 < \widehat{\mathbb{Q}}^{(2)}(1/X_T = 0)$, despite $\mathbb{Q}^{(1)}$ and $\mathbb{Q}^{(2)}$ being equivalent. $\square$

Remark 2 (Lack of martingale property in Proposition 4). One might wonder whether (i) of Proposition 4 can be strengthened by replacing each "local martingale" by "martingale." The last example illustrates that this cannot be done, not even in the case of X being a strictly positive, true $\mathbb{Q}$-martingale. To see this, we replace $\mathbb{Q}$ by $\mathbb{Q}^{(1)}$ and $\widehat{\mathbb{Q}}$ by $\mathbb{Q}^{(2)}$ and replace the processes N by X and X by $\omega^{(2)}$ in Proposition 4. Then N is a true $\mathbb{Q}^{(1)}$-martingale but $\tilde{N} = N$ is only a strict $\mathbb{Q}^{(2)}$-local martingale. $\square$

3 Change of numéraire and superreplication

In this section, we discuss a generalized change of numéraire that includes the case of nonnegative strict local martingale dynamics for the new numéraire ("Euro"). The new probability measure

($\mathbb{Q}^{€}$) is not necessarily equivalent to the old one ($\mathbb{Q}^{\$}$). For a given contingent claim, we then compute the corresponding cost of its superreplication under the original measure and under the new measure simultaneously.

3.1 Motivation

In the case of continuous asset price processes, the Fundamental Theorem of Asset Pricing states that a model is free of arbitrage if and only if there exists a probability measure that is equivalent to the physical probability measure, such that the asset prices have local martingale dynamics under this new measure; see Delbaen and Schachermayer (1994). From this point of view, it is legitimate to consider strict local martingales as models for prices of liquidly traded assets.

Indeed, a class of strict local martingales, the so-called "quadratic normal volatility" models, are being used to model exchange rates in the financial industry, mainly since they describe well features of observed asset prices and since they are easily analytically tractable; see also our companion paper Carr et al. (2012). Another example is the log-normal SABR model; if its asset price process is positively correlated with the stochastic volatility process then it follows not any more martingale dynamics but only local martingale dynamics; see Example 6.1 in Henry-Labordère (2009).

There is a price to pay if one uses strict local martingales instead of true martingales while modelling asset prices. The modelled prices display features that, on the first look, seem to imply obvious arbitrage opportunities and thus apparently contradict the Fundamental Theorem of Asset Pricing. To see this, denote by X a nonnegative, strict local martingale representing some asset price process, modelled under the risk-neutral measure $\mathbb{Q}^{\$}$ in a complete market. Then, over a sufficiently long time horizon $T > 0$, the strict inequality $\mathbb{E}^{\mathbb{Q}^{\$}}[X_T] < x_0$ holds. However, the expectation represents the minimal superreplicating price, so that the terminal value of X_T can be replicated for initial cost less than x_0; that is, a buy-and-hold strategy can be outperformed.

Furthermore, put-call parity fails if the underlying is modelled as a strict local martingale. Both of these observations seem to yield an arbitrage opportunity; for instance, sell the stock, obtain x_0 Dollars and invest $\mathbb{E}^{\mathbb{Q}^{\$}}[X_T]$ Dollars in the replicating portfolio. Then, at time T the positions have canceled out and the strictly positive difference $x_0 - \mathbb{E}^{\mathbb{Q}^{\$}}[X_T]$ is the arbitrage profit. However, this argument is not valid since the alleged arbitrage strategy is inadmissible, leading to a wealth process that cannot be bounded from below, and thus cannot be executed. More details on this argument are discussed in Ruf (2012b).

To sum up, on the one side, strict local martingales as models for asset prices yield arbitrage-free dynamics that can easily be calibrated and imply contingent claim prices that can be analytically computed; on the other side, minimal replicating prices in such models do not reflect our economic understanding of financial markets since current market prices are higher than the corresponding minimal superreplicating prices, and even do not agree with market conventions such as put-call parity.

One way out of this dilemma is to reconsider the notion of a contingent claim price, usually defined as the minimal cost to superreplicate the payoff of a contingent claim via dynamic trading. We propose to use the minimal cost for superreplicating a given contingent claim under two measures simultaneously as a pricing operator for contingent claim. In the case of Foreign Exchange markets with X modelling the exchange rate (as done before, we shall rely on the example of X

being the price of one Euro in Dollars) the two measures can be thought of as a "Dollar measure" $\mathbb{Q}^{\$}$ and a "Euro measure" $\mathbb{Q}^{€}$ corresponding to the choice of Dollars or Euros as numéraires. Since the two measures are not equivalent if X is a strict local martingale, as we will discuss in Subsection 3.2, the cost for superreplation increases since there are now more events in which the contingent claim has to be superreplicated. The inclusion of these additional cost in the definition of a pricing operator exactly restores put-call parity and yields international put-call equivalence as postulated in Giddy (1983) and a price for X_T at time zero that exactly equals x_0; see Subsection 3.3.

In particular within the Foreign Exchange context, considering both measures makes sense from an economic perspective. If there is a risk-neutral measure for the American investor $\mathbb{Q}^{\$}$ there also should be one for the European investor, and vice versa. The mathematical tools of constructing one from the other one in a consistent manner have been the content of Section 2. Then, from a superreplicating point of view it would be sensible to find a trading strategy that guarantees the terminal payoff under both measures $\mathbb{Q}^{\$}$ and $\mathbb{Q}^{€}$, both for the American and European investor.

One might ask for a reason why the Dollar and the Euro measures $\mathbb{Q}^{\$}$ and $\mathbb{Q}^{€}$ are not equivalent if the exchange rate X (or $1/X$) follows strict local martingale dynamics under $\mathbb{Q}^{\$}$ (or $\mathbb{Q}^{€}$). There is of course the technical argument of Section 2, which shows that it is necessary to consider events that have positive probability under only one of the two measures when dealing with strict local martingales. Each of these events represents an explosion of either X or $1/X$. It is clear that under a no-arbitrage assumption such an explosion can only be seen by one of the two measures; for example, an American investor cannot see the price of one Euro going to infinity since otherwise this would contradict the existence of the risk-neutral measure $\mathbb{Q}^{\$}$ corresponding to the Dollar as numéraire, which under such an explosion would become worthless.

To elaborate further on this, we take a slightly different point of view. Assume that there exists a physical probability measure under which both currencies Dollar and Euro have a positive probability of complete devaluation (hyperinflation). This corresponds to X and $1/X$ possibly hitting zero and infinity or vice versa. It is obvious that there cannot exist an equivalent probability measure under which X or $1/X$ have local martingale dynamics. To tackle this problem, we introduce two fictitious American and European investors who decide to neglect a Dollar hyperinflation $H^{\$} := \{X_T = 0\}$ or a Euro hyperinflation $H^{€} := \{1/X_T = 0\}$ and assign each zero probability to the corresponding event. Their physical measures then are $\mathbb{P}^{\$}(\cdot) := \mathbb{P}(\cdot | H^{\$^C})$ and $\mathbb{P}^{€}(\cdot) := \mathbb{P}(\cdot | H^{€^C})$. Under completeness, no-arbitrage and mild consistency conditions on these two measures the pricing operator we suggest yields precisely the cost of a trading strategy that uses both Dollars and Euros and that superreplicates some contingent claim under both measures $\mathbb{P}^{\$}$ and $\mathbb{P}^{€}$, and thus also $\mathbb{P}$-almost surely. This will be further discussed in Subsection 3.3 and Section 4.

We remark that Delbaen and Schachermayer (1995b) work out the connection of strict local martingales and changes of numéraires. While we understand a change of numéraire as a combination of a change of currency and the corresponding change of measure they start by looking at the change of currency only. Their results imply that, in an arbitrage-free model, a change of currency leads to the existence of arbitrage if the corresponding exchange rate is a strict local martingale under a unique risk-neutral measure. This follows from the observation that the money market in the first currency is replaced by the money market in the second currency. We circumvent the appearance of arbitrage here by associating the change of numèraire with the introduction of a

new probability measure that is not equivalent to the old one. It is exactly this lack of equivalence which avoids the arbitrage after the change of currency.

3.2 Change of numéraire

The technique of changing the numéraire in order to simplify computations has been introduced in the seminal paper of Geman et al. (1995). The underlying idea of this technique is that a change of units of the asset prices corresponds to an equivalent change of measure. For example, the price process S of an asset can be expressed in units of Dollars, but also in terms of another asset price $\tilde{S}$. If S has martingale dynamics under some measure $\mathbb{Q}$, then $S/\tilde{S}$, that is, the price process expressed in units of $\tilde{S}$, will have martingale dynamics under some measure $\tilde{\mathbb{Q}}$, which is equivalent to $\mathbb{Q}$.

This trick of changing the numéraire often simplifies the computation of contingent claim prices. It is particularly utilized for pricing contingent claims in Foreign Exchange markets, where it is often very useful to express asset prices in different currencies. For example, to change the price S of some future payoff from Dollars into Euros one divides it by the Dollar price $\tilde{S}$ of one Euro.

We illustrate that for changing the numéraire it is sufficient if the numéraire is only a nonnegative local martingale X under some probability measure $\mathbb{Q}^{\$}$. The following result is a corollary of the statements of Section 2 and generalizes an observation in Ruf (2011), where a similar result was derived under the additional assumption $\mathbb{Q}^{\$}(X_T > 0) = 1$:

Corollary 1 (Change of numéraire)**.** *There exists a probability measure $\mathbb{Q}^{€}$ with the properties of Theorem 1 such that for any $\mathcal{F}^0_T$-measurable random variable $Y \geq 0$ we have the following change of numéraire formula:*

$$\mathbb{E}^{\mathbb{Q}^{\$}}\left[Y\mathbf{1}_{\{X_T>0\}}\right] = x_0\mathbb{E}^{\mathbb{Q}^{€}}\left[\frac{Y}{X_T}\mathbf{1}_{\{1/X_T>0\}}\right].$$

Furthermore, let N denote some continuous $\mathbb{Q}^{\$}$-local martingale. Then, N/X is a $\mathbb{Q}^{€}$-local martingale on $[0, R)$ that can be uniquely extended to a continuous $\mathbb{Q}^{€}$-local martingale on $[0, R]$ with dynamics described in (8).

Proof. The first part of the statement follows directly from (3) with $\tau = T$ and Y replaced by Y/X_T. Propositions 4 and 1 then yield the statement. □

When modelling asset price dynamics, we now can start by specifying them under the $\mathbb{Q}^{\$}$-measure. For example, if the American investor, under the corresponding Dollar measure $\mathbb{Q}^{\$}$, sees the price process S of some asset, modelled as a $\mathbb{Q}^{\$}$-local martingale, then the European investor sees the price process S/X. This price process can be consistently modelled as a $\mathbb{Q}^{€}$-local martingale, even if X is a strict $\mathbb{Q}^{\$}$-local martingale. By (iii) of Proposition 3, this is of course trivial if N is a $\mathbb{Q}^{\$}$-martingale on $[0, S)$. Then S/X is a $\mathbb{Q}^{€}$-local martingale, hitting zero at the stopping time R (and staying there afterwards). However, if S is only a $\mathbb{Q}^{\$}$-local martingale, then things complicate, but can be solved by an application of Proposition 1, which yields a process $\tilde{S}$ that extends S/X to the interval $[0, R]$ and can be defined arbitrary as a $\mathbb{Q}^{€}$-local martingale afterwards, corresponding to the paths which only the Euro investor can see but not the Dollar

investor. Of course, this model construction can be reversed by first specifying the $\mathbb{Q}^{€}$-dynamics of the asset prices. Then, their $\mathbb{Q}^{\$}$-dynamics are fixed on $[0, S]$ and can in a second step be extended to $[0, T]$.

In the discussion so far we have implicitly assumed that the foreign interest rates are zero, to wit, there are no "dividends" payed out when holding a unit of the asset X, which, in this case, has local martingale dynamics. We refer the reader to Jarrow and Protter (2011), where the distinction is made between "bubbles" of the exchange rate itself and bubbles of the foreign money market account. Under zero interest rate as here, both kinds of bubbles are identical.

3.3 Economy and superreplication

We now are ready to introduce our model for the economy. For some $d \geq 1$ we assume the existence of $d+1$ tradable assets with continuous $\mathbb{F}^0$-progressively measurable price processes which under $\mathbb{Q}^{\$}$ are denoted by $S^{\$,(0)}, S^{\$,(1)}, \ldots, S^{\$,(d)}$. We set $S^{\$,(0)} \equiv 1$ and $S^{\$,(1)} = X$; to wit, $S^{\$,(0)}$ corresponds to a Dollar-denoted money market which pays zero interest rate and $S^{\$,(1)}$ corresponds to the Dollar price of one Euro. We assume that for $i = 0, \ldots, d$, the process $S^{\$,(i)}$ have local martingale dynamics under the probability measure $\mathbb{Q}^{\$}$, which directly excludes arbitrage in the sense of Delbaen and Schachermayer (1994).

As we discussed in Subsection 3.2, there exists a probability measure $\mathbb{Q}^{€}$, which corresponds to the probability measure with X as numéraire. Under this measure, the processes $S^{€,(i)} := S^{\$,(i)}/X$ for $i = 0, \ldots, d$ are local martingales, at least up to some stopping time R, which is the first time that the process $1/X$ hits zero. We always assume that the processes $S^{€,(i)}$ correspond to the extension of Corollary 1; in particular, we set $S^{€,(1)} \equiv 1$. The processes $S^{€,(i)}$ correspond to the price of different assets, denoted in Euros. For instance, $S^{€,(0)}$ corresponds to the Euro price of one Dollar and $S^{€,(1)}$ corresponds to the Euro-denoted money market, which also pays zero interest rate.

The two probability measures $\mathbb{Q}^{\$}$ and $\mathbb{Q}^{€}$ are not necessarily equivalent with respect to each other. This implies that the $\mathbb{Q}^{€}$-dynamics of the Euro price processes $S^{€,(i)}$ are only fully determined by the corresponding $\mathbb{Q}^{\$}$-dynamics $S^{\$,(i)}$ if these are true $\mathbb{Q}^{\$}$-martingales or if X is a true $\mathbb{Q}^{\$}$-martingale; see Proposition 3. Thus, we have to assume that the price processes $S^{€,(i)}$ have local martingale dynamics also after the stopping time R. This means that we assume that there exists no arbitrage opportunity for the European investor.

Remark 3 (A note on stochastic integration). In the following, we shall need stochastic integrals with respect to $S^{\$,(i)}$ and $S^{€,(i)}$. Stochastic integrals are usually only constructed on complete probability spaces; see Protter (2003) and the discussion on page 97 of Stroock and Varadhan (2006). However, in order to construct the measure $\mathbb{Q}^{€}$ in Theorem 1, we explicitly had to abstain from augmenting the underlying filtration $\mathbb{F}^0$ with the subsets of $\mathbb{Q}^{\$}$-nullsets.

One way out is to assume that for $i = 0, \ldots, d$ the continuous processes $S^{\$,(i)}$ and $S^{€,(i)}$ are Itô processes under the corresponding measure. To wit, we assume that their quadratic covariations $\langle S^{\$,(i)} \rangle$ and $\langle S^{€,(i)} \rangle$ are absolutely continuous with respect to the Lebesgue measure; that is, $\langle S^{\$,(i)} \rangle_t$ and $\langle S^{€,(i)} \rangle_t$ are absolutely continuous functions of t. This, for example, is satisfied if the local martingales $S^{\$,(i)}$ and $S^{€,(i)}$ can be represented as solutions to stochastic differential equations with a Brownian motion as driving force. Then, we can rely on the construction of the stochastic integral in Theorem 4.3.6 and Exercise 4.6.11 of Stroock and Varadhan (2006). This

construction does not require the filtration to be complete.

Another approach, now without the assumption of having an Itô process, is to construct the stochastic integral with respect to the temporarily completed filtration and then take a continuous version of this integral process, progressively measurable with respect to $\mathbb{F}^0$, whose existence follows from Lemma 2.1 in Soner et al. (2011). This needs to be done once under the $\mathbb{Q}^{\$}$- and once under the $\mathbb{Q}^{€}$-measure. $\square$

Stochastic integration is used in the following definition:

Definition 2 (Trading strategy). A *trading strategy* is an $\mathbb{R}^{d+1}$-valued, $\mathbb{F}^0$-progressively measurable process η such that

- we have
$$\mathbb{Q}^{\$}\left(\int_0^T \eta_t^{\mathsf{T}} \langle S^{\$}\rangle_t \eta_t \mathrm{d}t < \infty\right) = 1 = \mathbb{Q}^{€}\left(\int_0^T \eta_t^{\mathsf{T}} \langle S^{€}\rangle_t \eta_t \mathrm{d}t < \infty\right),$$
where $\langle S^{\$}\rangle$ (respectively, $\langle S^{€}\rangle$) denotes the $(d+1)\times(d+1)$ matrix-valued quadratic variation process of the components of $S^{\$}$ (respectively, $S^{€}$),

- its corresponding Dollar wealth process $V^{\$,\eta} = (V_t^{\$,\eta})_{t\in[0,T]}$ and Euro wealth process $V^{€,\eta} = (V_t^{€,\eta})_{t\in[0,T]}$ defined by
$$V_t^{\$,\eta} := \sum_{i=0}^d \eta_t^{(i)} S_t^{\$,(i)}, \tag{11}$$
$$V_t^{€,\eta} := \sum_{i=0}^d \eta_t^{(i)} S_t^{€,(i)} \tag{12}$$
for all $t \in [0,T]$ stay nonnegative almost surely under the corresponding measure $\mathbb{Q}^{\$}$ and $\mathbb{Q}^{€}$, respectively, and

- the self-financing condition holds, that is,
$$\mathrm{d}V_t^{\$,\eta} = \sum_{i=0}^d \eta_t^{(i)} \mathrm{d}S_t^{\$,(i)},$$
$$\mathrm{d}V_t^{€,\eta} = \sum_{i=0}^d \eta_t^{(i)} \mathrm{d}S_t^{€,(i)}$$
for all $t \in [0,T]$, where the dynamics are computed under the corresponding measure $\mathbb{Q}^{\$}$ and $\mathbb{Q}^{€}$, respectively.

We shall say that η is a trading strategy for initial capital $v \in [0,\infty)$ expressed in Dollars if
$$v = V_0^{\$,\eta} = \sum_{i=0}^d \eta_0^{(i)} S_0^{\$,(i)}$$
holds; and similarly for initial capital v expressed in Euros. $\square$

Thus, at any time $t \in [0,T]$ each component of η_t determines the current number of shares of each asset held at that point of time. Clearly, the nonnegative condition on $V^{\$,\eta}$ implies the one on $V^{€,\eta}$, but only up to the stopping time R. Proposition 3 implies that $V^{\$,\eta}/X$ is a $\mathbb{Q}^{€}$-local martingale on $[0,R)$; indeed a simple application of Itô's rule yields that $V_\tau^{€,\eta} = V_\tau^{\$,\eta}/X_\tau$ $\mathbb{Q}^{€}$- and $\mathbb{Q}^{\$}$-almost surely for all stopping times τ with $\mathbb{Q}^{€}(\tau < R \wedge S) = 1$. The same computations show that the self-financing condition under $\mathbb{Q}^{\$}$ implies the one under $\mathbb{Q}^{€}$ if $\mathbb{Q}^{€}(R = T+1) = 1$, that is, if X is a true martingale; see Geman et al. (1995).

We call any pair of nonnegative random variables $(D^{\$}, D^{€})$ a *contingent claim* if both random variables are measurable with respect to $\mathcal{F}_T^0$ and satisfy $D^{€} = D^{\$}/X_T$ on the event $\{0 < X_T < \infty\}$. The random variable $D^{\$}$ ($D^{€}$) corresponds to the Dollar (Euro) price of a contingent claim, as seen by the American (European) investor. We remind the reader that the event $\{X_T = 0\}$ has zero $\mathbb{Q}^{€}$-probability, but might have positive $\mathbb{Q}^{\$}$-probability, and a similar statement holds for the event $\{1/X_T = 0\}$.

We represent a contingent claim as a pair of random variables in order to be able to exactly express its payoff both in Dollars and in Euros including in the event of X hitting infinity. For example, the contingent claim $(X_T, 1)$ pays off one Euro at maturity, the contingent claim $(X_T, \mathbf{1}_{\{1/X_T > 0\}})$ pays off one Euro if the price of one Euro in Dollars did not explode. For some $K \in \mathbb{R}$, the claims $D_K^{C,\$} := ((X_T - K)^+, (1 - K/X_T)^+)$ and $D_K^{P,\$} := ((K - X_T)^+, (K/X_T - 1)^+)$ are called *call* and *put*, respectively, on one Euro with strike K and maturity T. Equivalently, by exchanging X_T with $1/X_T$, we define calls and puts on one Dollar and denote them by $D_K^{C,€}$ and $D_K^{P,€}$. In Foreign Exchange markets, *self-quantoed calls* are traded, defined as $D_K^{SQC,\$} := X_T D_K^{C,\$} = (X_T(X_T - K)^+, (X_T - K)^+)$ for some $K \in \mathbb{R}$.

We shall assume that the market is *complete* both for the American investor and for the European investor; for any contingent claim $(D^{\$}, D^{€})$ with $p^{\$} = \mathbb{E}^{\mathbb{Q}^{\$}}[D^{\$}] < \infty$ and $p^{€} = \mathbb{E}^{\mathbb{Q}^{€}}[D^{€}] < \infty$, there exist some trading strategy $\eta^{\$}$ for initial capital $p^{\$}$ and some trading strategy $\eta^{€}$ for initial capital (expressed in Euros) $p^{€}$ such that

$$\mathbb{Q}^{\$}\left(V_T^{\$,\eta^{\$}} = D^{\$}\right) = 1 = \mathbb{Q}^{€}\left(V_T^{€,\eta^{€}} = D^{€}\right).$$

The replicability of any contingent claim under $\mathbb{Q}^{\$}$ does not necessarily imply that any contingent claim can be replicated under $\mathbb{Q}^{€}$ since, in general, the two measures are not equivalent. However, we have the following observation:

Proposition 5 (Completeness)**.** *All $\mathcal{F}_S^0$-measurable contingent claims can be replicated under $\mathbb{Q}^{\$}$ if and only if all $\mathcal{F}_R^0$-measurable contingent claims can be replicated under $\mathbb{Q}^{€}$.*

Proof. Let us assume that $\mathbb{Q}^{\$}$ is the unique local martingale measure on $\mathcal{F}_S^0$; to wit, any probability measure $\widehat{\mathbb{Q}}^{\$}$ under which $S^{\$,(i)}$ are local martingales for $i = 0, \dots, d$ and which is equivalent to $\mathbb{Q}^{\$}$ equals $\mathbb{Q}^{\$}$ on $\mathcal{F}_S^0$. It is sufficient to show that any probability measure $\widehat{\mathbb{Q}}^{€}$ that is equivalent to $\mathbb{Q}^{€}$ and under which the asset prices $S^{€,(i)}$ are local martingales for $i = 0, \dots, d$ satisfies $\widehat{\mathbb{Q}}^{€}|_{\mathcal{F}_R^0} = \mathbb{Q}^{€}|_{\mathcal{F}_R^0}$. Assume that the two measures differ. Then, there exists some strictly positive $\mathbb{Q}^{€}$-martingale Z such that $Z_T^R \not\equiv 1$ and $Z^R S^{€,(i)R}$ are $\mathbb{Q}^{€}$-local martingales for $i = 0, \dots, d$. Since $\bigcup_{i=1}^\infty \mathcal{F}_{R_i}^0$ forms a pi-system and generates $\mathcal{F}_R^0$ and since $\lim_{i \to \infty} S_i = T + 1$ $\mathbb{Q}^{€}$-almost surely, there exists some $j \in \mathbb{N}$ such that $Z_T^{R_j \wedge S_j} \not\equiv 1$.

On $\mathcal{F}^0_{R_j\wedge S_j}$, the two probability measures $\mathbb{Q}^{\$}$ and $\mathbb{Q}^{€}$ are equivalent to each other such that $Z^{R_j\wedge S_j} \equiv (Z^{R_j\wedge S_j} S^{€,(0)^{R_j\wedge S_j}}) \cdot X^{R_j\wedge S_j}$ is a $\mathbb{Q}^{\$}$-martingale. The probability measure $\widehat{\mathbb{Q}}^{\$}$ defined by $\mathrm{d}\widehat{\mathbb{Q}}^{\$}/\mathrm{d}\mathbb{Q}^{\$} = Z_T^{R_j\wedge S_j}$ is equivalent to but different from $\mathbb{Q}^{\$}$. Furthermore, it can easily been seen that the dollar price processes $S^{\$,(i)}$ are $\widehat{\mathbb{Q}}^{\$}$-local martingales for $i = 0, \ldots, d$, contradicting the uniqueness of the local martingale measure $\mathbb{Q}^{\$}$. The other direction follows in the same way. □

Example 3 (Lack of equivalence (continued)). We slightly extend the discussion of Example 2. There, we basically have illustrated that the sets of equivalent martingale measures indexed over Radon-Nikodym derivatives do not necessarily agree for $\mathbb{Q}^{\$}$ and $\mathbb{Q}^{€}$. In particular, if in this example we assume $d = 1$, that is, only X is traded and if we set $\mathbb{Q}^{\$} = \mathbb{Q}^{(1)}$ and $\mathbb{Q}^{€} = \widehat{\mathbb{Q}}^{(1)}$, then $w^{(2)}$ generates only an equivalent local martingale measure if applied to $\mathbb{Q}^{\$}$ but not if applied to $\mathbb{Q}^{€}$.

However, Proposition 5 states that there must also exist probability measures equivalent to $\mathbb{Q}^{€}$ under which $1/X$ is a local martingale. And indeed, stopping $w^{(2)}$ when hitting any level greater than 1 yields such a Radon-Nikodym derivative. □

Remark 4 (A seeming paradox). Let the exchange rate X be a strict local martingale hitting zero with positive probability under $\mathbb{Q}^{\$}$. If one assumes that a European investor neither has any arbitrage opportunities, then there exists a risk-neutral measure that equals $\mathbb{Q}^{€}$ of Corollary 1. Under this measure, $1/X$ is again a strict local martingale. Then, we have the following paradox. Under the Dollar measure, one can replicate the payoff of one Euro for less than one Euro; simultaneously, under the European measure, one can replicate the payoff of one Dollar for less than one Dollar. To conclude, the exchange rate reflects an overly-high price (compared to their replicating cost) both for the Dollar and for the Euro; thus being at the same time too high and too low for the Dollar!

Clearly, this paradox can be easily explained by reminding oneself that the two measures $\mathbb{Q}^{\$}$ and $\mathbb{Q}^{€}$ are not equivalent; and therefore the investors are concerned with different events when replicating a Euro or a Dollar, respectively. However, the paradox illustrates that by means of the classical theory only, there cannot be a consistent no-arbitrage theory under different currencies if strict local martingales are involved. □

The next theorem constitutes the core result of this section:

Theorem 2 (Minimal superreplicating price)**.** *The minimal joint $\mathbb{Q}^{\$}$- and $\mathbb{Q}^{€}$-superreplicating price $p^{\$}(D)$ ($p^{€}(D)$) for a contingent claim $D = (D^{\$}, D^{€})$ is, expressed in Dollars (Euros),*

$$p^{\$}(D) = \mathbb{E}^{\mathbb{Q}^{\$}}\left[D^{\$}\right] + x_0 \mathbb{E}^{\mathbb{Q}^{€}}\left[D^{€}\mathbf{1}_{\{1/X_T=0\}}\right], \tag{13}$$
$$p^{€}(D) = \mathbb{E}^{\mathbb{Q}^{€}}\left[D^{€}\right] + \frac{1}{x_0}\mathbb{E}^{\mathbb{Q}^{\$}}\left[D^{\$}\mathbf{1}_{\{X_T=0\}}\right]] = \frac{p^{\$}(D)}{x_0}.$$

More precisely, there exists some trading strategy η for initial capital $p^{\$}(D)$ (expressed in Dollars) such that

$$\mathbb{Q}^{\$}\left(V_T^{\$,\eta} = D^{\$}\right) = 1 = \mathbb{Q}^{€}\left(V_T^{€,\eta} = D^{€}\right); \tag{14}$$

and there exists no $\tilde{p} < p^{\$}(D)$ and no trading strategy $\tilde{\eta}$ for initial capital $\tilde{p}$ (expressed in Dollars) such that (14) *holds with η replaced by $\tilde{\eta}$.*

Proof. Since the market is assumed to be complete there exist trading strategies ν for initial capital $p^{(1)} := \mathbb{E}^{\mathbb{Q}^\$}[D^\$]$ and θ for initial capital (expressed in Euros) $p^{(2)} := \mathbb{E}^{\mathbb{Q}^€}[D^€ \mathbf{1}_{\{1/X_T=0\}}]$ such that $V^{\$,\nu}$ is a $\mathbb{Q}^\$$-martingale, $V^{€,\theta}$ a $\mathbb{Q}^€$-martingale, and

$$\mathbb{Q}^\$ \left(V_T^{\$,\nu} = D^\$ \right) = 1 = \mathbb{Q}^€ \left(V_T^{€,\theta} = D^€ \mathbf{1}_{\{1/X_T=0\}} \right).$$

We now prove that $\eta := \nu + \theta$ replicates $D^\$$ under $\mathbb{Q}^\$$ and $D^€$ under $\mathbb{Q}^€$. We start by observing that $V_T^{\$,\eta} = V_T^{\$,\nu} + V_T^{\$,\theta}$. Now, $V^{€,\theta}$ is a $\mathbb{Q}^€$-martingale, thus $V^{\$,\theta} = XV^{€,\theta}$ is a $\mathbb{Q}^\$$-local martingale by (iv) of Proposition 3 with $V_S^{\$,\theta} = 0$ on $\{S \leq T\}$. This in conjunction with (4) shows that $\mathbb{Q}^\$(V_T^{\$,\theta} = 0) = 1$. Thus, we have shown that η replicates D under $\mathbb{Q}^\$$.

We observe that under $\mathbb{Q}^€$

$$V_\tau^{€,\nu} = \frac{1}{X_\tau} X_\tau V_\tau^{€,\nu} = \frac{1}{X_\tau} V_\tau^{\$,\nu}$$

for all stopping times τ with $\mathbb{Q}^€(\tau < R) = 1$. Now, (iii) of Proposition 3 yields that $V^{€,\nu}$ is a local martingale on $[0, R)$ with unique extension $V_R^{€,\nu} = 0$. Thus,

$$\begin{aligned} V_T^{€,\nu+\theta} &= V_T^{€,\nu} + V_T^{€,\theta} \\ &= \frac{1}{X_T} V_T^{\$,\nu} \mathbf{1}_{\{R=T+1\}} + D^€ \mathbf{1}_{\{1/X_T=0\}} \\ &= \frac{1}{X_T} D^\$ \mathbf{1}_{\{1/X_T>0\}} + D^€ \mathbf{1}_{\{1/X_T=0\}} \\ &= D^€, \end{aligned}$$

which implies that $\nu + \theta$ replicates $D^€$ under $\mathbb{Q}^€$. The initial cost for the strategy η is, expressed in Dollars, exactly $p^\$ = p^{(1)} + x_0 p^{(2)}$. The second identity follows in the same manner.

Now, let $\tilde{p} \in \mathbb{R}_+$ and $\tilde{\eta}$ be a trading strategy for initial capital $\tilde{p}$ (expressed in Dollars) that superreplicates $D^\$$ under $\mathbb{Q}^\$$ and $D^€$ under $\mathbb{Q}^€$. Then, $\tilde{p} = M_0 + N_0$, where M and N are the martingale and strict local martingale part of the Riesz decomposition $V^{\$,\tilde{\eta}} = M + N$ under $\mathbb{Q}^\$$ with $\mathbb{Q}^\$(N_T = 0) = 1$; to wit, $M_t = \mathbb{E}^{\mathbb{Q}^\$}[V_T^{\$,\tilde{\eta}} | \mathcal{F}_t^0]$ and $N_t := V_t^{\$,\tilde{\eta}} - M_t$ for all $t \in [0, T]$; see Theorem 2.3 of Föllmer (1973) for the case of a not completed filtration. Since $\tilde{\eta}$ superreplicates $D^\$$ under $\mathbb{Q}^\$$ we obtain $M_0 \geq \mathbb{E}^{\mathbb{Q}^\$}[D^\$]$. As in the first part of the proof, we have $\mathbb{Q}^€(\{M_T > 0\} \bigcap \{R \leq T\}) = 0$. Thus, the superreplication of $D^€$ under $\mathbb{Q}^€$ by $\tilde{\eta}$ implies that $\mathbb{Q}^€(\tilde{N}_T \geq D^€ \mathbf{1}_{\{1/X_T=0\}}) = 1$, where $\tilde{N}$ is the extension of the $\mathbb{Q}^€$-local martingale N/X on $[0, R)$; see (ii) of Proposition 3 and Proposition 1. This implies $N_0 \geq x_0 \mathbb{E}^{\mathbb{Q}^€}[D^€ \mathbf{1}_{\{1/X_T=0\}}]$ yielding $\tilde{p} \geq p^\$(D)$. $\square$

The last theorem yields the smallest amount of Dollars (Euros) that is needed to superreplicate a claim D under both measures $\mathbb{Q}^\$$ and $\mathbb{Q}^€$. The corresponding replicating strategy is, as the proof illustrates, a sum of two components. The first component is the standard strategy that replicates the claim under one of the two measures; the second component replicates the claim under the events that only the other measure can "see."

The next few corollaries are direct implications of the last theorem. We usually formulate them only in terms of the Dollar pricing operator $p^\$$ but it is clear that they also hold for the Euro pricing operator $p^€$.

Corollary 2 (Linearity of pricing operator)**.** *The pricing operator* $p^\$$ *of* (13) *is linear; for any claims* $D_1 = (D_1^\$, D_1^€), D_2 = (D_2^\$, D_2^€)$ *and* $a \geq 0$ *we have*

$$p^\$(D_1 + aD_2) = p^\$(D_1) + ap^\$(D_2),$$

where $D_1 + aD_2 := (D_1^\$ + aD_2^\$, D_1^€ + aD_2^€)$.

Proof. The statement follows directly from the linearity of expectations. □

Corollary 3 (Strict local martingality of wealth process)**.** *The wealth process* $V^{\$,\eta}$ *of Theorem 2 is a* $\mathbb{Q}^\$$*-local martingale. It is a strict* $\mathbb{Q}^\$$*-local martingale if and only if* X *is a strict* $\mathbb{Q}^\$$*-local martingale and* $\mathbb{Q}^€(\{D^€ > 0\} \bigcap \{1/X_T = 0\}) > 0$*. Similarly, the wealth process* $V^{€,\eta}$ *is a* $\mathbb{Q}^€$*-local martingale. It is a strict* $\mathbb{Q}^€$*-local martingale if and only if* $1/X$ *is a strict* $\mathbb{Q}^€$*-local martingale and* $\mathbb{Q}^\$(\{D^\$ > 0\} \bigcap \{X_T = 0\}) > 0$*.*

Proof. The local martingality of the wealth processes under the corresponding measures follows directly from their definition in (11) and (12). The strict local martingality follows from checking when $p^\$(D)$ of (13) satisfies $p < E^{\mathbb{Q}^\$}[D^\$]$ and $p^\$(D) < E^{\mathbb{Q}^€}[D^€]$, respectively. □

Corollary 4 (Price of a Euro)**.** *The minimal joint* $\mathbb{Q}^\$$*- and* $\mathbb{Q}^€$*-superreplicating price of* $(X_T, 1)$ *is* x_0 *(expressed in Dollars) or* 1 *(expressed in Euros).*

Proof. The statement follows from (3) which implies that $x_0\mathbb{Q}^€(1/X_T = 0) = x_0 - E^{\mathbb{Q}^\$}[X_T]$. □

Clearly, the corresponding replicating strategy is the buy-and-hold strategy of one Euro.

Corollary 5 (Put-call parity)**.** *The prices of puts and calls simplify under the pricing operator* $p^\$$ *to*

$$\begin{aligned} p^\$(D_K^{P,\$}) &= E^{\mathbb{Q}^\$}[(K - X_T)^+], \\ p^\$(D_K^{C,\$}) &= E^{\mathbb{Q}^\$}[(X_T - K)^+] + x_0\mathbb{Q}^€(1/X_T = 0); \end{aligned} \tag{15}$$

moreover, the put-call parity

$$p^\$(D_K^{C,\$}) + K = p^\$(D_K^{P,\$}) + x_0 \tag{16}$$

holds, where $K \in \mathbb{R}$ *denotes the strike of the call* $D_K^{C,\$}$ *and put* $D_K^{P,\$}$*.*

Proof. The statement follows directly from (13) and the linearity of expectation. □

We refer to Madan and Yor (2006) for alternative representations of the call price in (15).

Giddy (1983) introduces the notion of *international put-call equivalence* which relates the price of a call in one currency with the price of a put in the other currency; see also Grabbe (1983).

Corollary 6 (International put-call equivalence)**.** *The pricing operators* $p^\$$ *and* $p^€$ *satisfy international put-call equivalence:*

$$\begin{aligned} p^\$\left(D_K^{C,\$}\right) &= x_0 K p^€\left(D_{\frac{1}{K}}^{P,€}\right), \\ p^\$\left(D_K^{P,\$}\right) &= x_0 K p^€\left(D_{\frac{1}{K}}^{C,€}\right) \end{aligned}$$

for all $K > 0$*.*

Proof. We obtain

$$\begin{aligned}
x_0 K p^{€}\left(D^{P,€}_{\frac{1}{K}}\right) &= x_0 K \left(\mathbb{E}^{\mathbb{Q}^{€}}\left[\left(\frac{1}{K}-\frac{1}{X_T}\right)^{+}\mathbf{1}_{\{1/X_T>0\}}\right]+\mathbb{E}^{\mathbb{Q}^{€}}\left[\frac{1}{K}\mathbf{1}_{\{1/X_T=0\}}\right]\right)\\
&= x_0\left(\mathbb{E}^{\mathbb{Q}^{€}}\left[(X_T-K)^{+}\frac{1}{X_T}\mathbf{1}_{\{1/X_T>0\}}\right]+\mathbb{Q}^{€}\left(\frac{1}{X_T}=0\right)\right)\\
&= E^{\mathbb{Q}^{\$}}\left[(X_T-K)^{+}\mathbf{1}_{\{X_T>0\}}\right]+x_0\mathbb{Q}^{€}(1/X_T=0)\\
&= p^{\$}\left(D^{C,\$}_{K}\right),
\end{aligned}$$

where we have used the identities of Corollaries 1 and 5. The second equivalence follows in the same way or from the put-call parity in (16). □

The next remark discusses how our result motivates and generalizes Lewis' Generalized Pricing Formualas.

Remark 5 (Lewis' Generalized Pricing Formulas). Within Markovian stochastic volatiliy models, Lewis (2000) derives call and put prices which exactly correspond to (13) when applied to the call payoff $D^{C,\$}_K$ or put payoff $D^{P,\$}_K$. Lewis starts from the postulate that put-call-parity holds and then shows that the correction term that is added to the expected payoff under $\mathbb{Q}^{\$}$ corresponds to the probability of some process exploding under another measure (corresponding here to $\mathbb{Q}^{€}$). We here start from an economic argument by defining the price as the minimal superreplicating cost for a contingent claim under two, possibly non-equivalent measures that arise from a change of numéraire. We then show that this directly implies put-call parity. This approach also yields a generalization of Lewis' pricing formula to arbitrary, possibly path-dependent contingent claims. □

Example 4 (Singular measures II). We continue here our discussion of Example 1 with $\mathbb{Q}^{\$}=\mathbb{Q}$ and $\mathbb{Q}^{€}=\widehat{\mathbb{Q}}$. Although the exchange rate X is a $\mathbb{Q}^{\$}$-local martingale, from the classical point of view of a Dollar investor the minimal superreplicating price of one Euro at time T is zero because under $\mathbb{Q}^{\$}$ there are only paths under which this contingent claim becomes worthless. However, by means of the correction term, (13) yields a price $p^{\$}((X_T,1))=x_0$, when considering the minimal joint $\mathbb{Q}^{\$}$- and $\mathbb{Q}^{€}$-superreplicating price of one Euro.

For the self-quantoed call $D^{SQC,\$}_K$, the classical price would be again zero; however, considering also the paths that the European investor under $\mathbb{Q}^{€}$ can see, Theorem 2 suggests a price $p^{\$}(D^{SQC,\$}_K)=\infty$ since $\mathbb{E}^{\mathbb{Q}^{€}}[(X_T-K)^{+}]=\infty$. □

In many applications, however, the measures $\mathbb{Q}^{\$}$ and $\mathbb{Q}^{€}$ do not become singular. Often, one measure is absolutely continuous with respect to the other measure. In this case, the formula for computing p simplifies:

Corollary 7 (Absolutely continuous measures)**.** *If* $\mathbb{Q}^{\$}(S=T+1)=1$*, then* $p^{€}$ *can be computed as*

$$p^{€}((D^{\$},D^{€}))=\mathbb{E}^{\mathbb{Q}^{€}}[D^{€}].$$

If $\mathbb{Q}^{€}(R=T+1)=1$*, that is, if* X *is* $\mathbb{Q}^{\$}$*-martingale, then*

$$p^{\$}((D^{\$},D^{€}))=\mathbb{E}^{\mathbb{Q}^{\$}}[D^{\$}].$$

Proof. Assume that $\mathbb{Q}^{\$}(S = T+1) = 1$. Then, Remark 1 implies that $\mathbb{Q}^{\$}$ is absolutely continuous with respect to $\mathbb{Q}^{€}$. Thus, if a trading strategy superreplicates $D^{€}$ $\mathbb{Q}^{€}$-almost surely for an European investor, then it also superreplicates $D^{\$}$ $\mathbb{Q}^{\$}$-almost surely for an American investor. The second statement can be shown analogously; however, it follows also directly from (13). □

Example 5 (Reciprocal of a three-dimensional Bessel process). We set $n = d = T = 1$ and let X denote a nonnegative $\mathbb{Q}^{\$}$-local martingale identically distributed as the reciprocal of a three-dimensional Bessel process starting in 1; in particular, there exists a Brownian motion $W = (W_t)_{t \in [0,T]}$ such that

$$X_t = 1 + \int_0^t X_u^2 \mathrm{d}W_u$$

for all $t \in [0,T]$. It is well-known that X is strictly positive and that $1/X$ is distributed under $\mathbb{Q}^{€}$ as a Brownian motion stopped in zero; see for example Delbaen and Schachermayer (1995a). Since X is strictly positive, the discussion in Remark 1 yields that $\mathbb{Q}^{\$}$ is absolutely continuous with respect to $\mathbb{Q}^{€}$; thus, Corollary 7 applies.

Let us study the self-quantoed call $D_K^{SQC,\$}$. Since Brownian motion hits 0 in any time interval with positive probability we obtain that X hits ∞ with positive $\mathbb{Q}^{€}$-probability. This yields directly a minimal joint $\mathbb{Q}^{\$}$- and $\mathbb{Q}^{€}$-superreplicating price $p^{\$}(D_K^{SQC,\$}) = \infty$. It is interesting to note that, as in Example 4, the classical price is finite:

$$\begin{aligned}
E^{\mathbb{Q}^{\$}}[X_T(X_T - K)^+] &\leq E^{\mathbb{Q}^{\$}}[X_T^2] \\
&= \mathbb{E}^{\mathbb{Q}^{€}}[X_T \mathbf{1}_{\{1/X_T > 0\}}] \\
&= \frac{1}{\sqrt{2\pi T}} \int_0^\infty \frac{1}{y} \left(\exp\left(-\frac{(y-1)^2}{2T} \right) - \exp\left(-\frac{(y+1)^2}{2T} \right) \right) \mathrm{d}y \\
&< \infty
\end{aligned}$$

for all $K \geq 0$, where we have plugged in the density of killed Brownian motion; see Exercise III.1.15 in Revuz and Yor (1999). □

We remark that, as a corollary of Remark 1, in our setup there are only positive "bubbles" under the corresponding measure. A bubble is usually defined as the difference of the current price and the expectation of the terminal value of an asset; that is $x_0 - \mathbb{E}^{\mathbb{Q}^{\$}}[X_T]$ and $1/x_0 - \mathbb{E}^{\mathbb{Q}^{€}}[1/X_T]$, respectively, here. It is possible, that both bubbles are strictly positive; however, negative bubbles cannot occur by the supermartingale property of the asset price processes under the corresponding measure. This contrasts Jarrow and Protter (2011), where negative bubbles are discussed, however only when considering the Dollar measure $\mathbb{Q}^{\$}$, which is not the risk-neutral measure of a European investor.

In the next section, we provide an interpretation of a bubble (strict local martingality of the exchange rate) as the possibility of a hyperinflation under some dominating "real-world" measure $\mathbb{P}$. If for both currencies such hyperinflations have positive $\mathbb{P}$-probability, then there are positive bubbles in both X and $1/X$.

4 A physical measure

Upon now, we have started from a risk-neutral measure $\mathbb{Q}^{\$}$ of the American investor and then have constructed a risk-neutral measure $\mathbb{Q}^{€}$ of the European investor. In this section, we start by specifying a physical probability measure $\mathbb{P}$. To begin with, let $\mathbb{P}$ denote any probability measure on $(\Omega, \mathcal{F}_T^0)$, possibly with explosions of X, that is, complete devaluations (hyperinflations) of Dollars and Euros, defined as

$$H^{\$} := \{X_T = \infty\} = \{R \leq T\},$$
$$H^{€} := \{X_T = 0\} = \{S \leq T\},$$

are both allowed to have positive probability under $\mathbb{P}$ but one excludes the other. We shall conclude this section by discussing some empirical evidence that such events occur by reviewing the macroeconomic literature in Subsection 4.1.

If indeed the two events have positive probability there cannot exist a risk-neutral measure equivalent to $\mathbb{P}$ such that either X or $1/X$ follow local martingale dynamics. Let us instead introduce the two artificial measures

$$\mathbb{P}^{\$}(\cdot) := \mathbb{P}(\cdot | H^{\$^C}) = \mathbb{P}(\cdot | R = T+1), \tag{17}$$
$$\mathbb{P}^{€}(\cdot) := \mathbb{P}(\cdot | H^{€^C}) = \mathbb{P}(\cdot | S = T+1), \tag{18}$$

where we have conditioned the physical measure on the events $H^{\$^C}$ and $H^{€^C}$ that no hyperinflation occurs. We observe that both $\mathbb{P}^{\$}$ and $\mathbb{P}^{€}$ are absolutely continuous with respect to $\mathbb{P}$ and that $\mathbb{P}$ is absolutely continuous with respect to their average $(\mathbb{P}^{\$}(\cdot) + \mathbb{P}^{€}(\cdot))/2$ since both 0 and ∞ are absorbing states of X.

We shall assume that $\mathbb{P}^{\$}$ allows for an equivalent local martingale measure, which we denote by $\mathbb{Q}^{\$}$. We then can construct, as in Subsection 3.2, a probability measure $\mathbb{Q}^{€}$, under which $1/X$ follows local martingale dynamics. It is reasonable to assume that $\mathbb{Q}^{€}$ is equivalent to $\mathbb{P}^{€}$. However, as we will illustrate, this is not always true, even if $\mathbb{P}^{€}$ allows for an equivalent local martingale measure $\widehat{\mathbb{Q}}^{€}$. The next result clarifies the consistency conditions needed to be imposed on $\mathbb{P}$ in order to obtain such an equivalence:

Proposition 6 (Consistency conditions for the physical measure)**.** *Assume that*

(A) the American investor has no arbitrage opportunities under the probability measure $\mathbb{P}^{\$}$,

(B) the European investor has no arbitrage opportunities under the probability measure $\mathbb{P}^{€}$.

Then,

(i) the measures $\mathbb{P}^{\$}$ and $\mathbb{P}^{€}$ are equivalent on $\mathcal{F}^0_{R_i \wedge S_j}$ for all $i, j \in \mathbb{N}$; and

(ii) the measure $\mathbb{Q}^{\$}$ is the unique probability measure on $\mathcal{F}_S^0$ that is equivalent to $\mathbb{P}^{\$}$ and under which the processes $S^{\$,(i)}$ have local martingale dynamics for $i = 0, \ldots, d$,

if and only if the probability measure $\mathbb{Q}^{€}$, constructed in Corollary 1, is the unique probability measure on $\mathcal{F}_R^0$ equivalent to $\mathbb{P}^{€}$ such that the processes $S^{€,(i)}$ have local martingale dynamics for $i = 0, \ldots, d$.

Proof. If $\mathbb{Q}^{€}$ is the unique local martingale measure equivalent to $\mathbb{P}^{€}$, then (i) directly follows from the equivalence of $\mathbb{Q}^{\$}$ and $\mathbb{Q}^{€}$ on $\mathcal{F}^0_{R_i \wedge S_j}$ and (ii) is proven in Proposition 5. For the other direction, we observe that there exists, by the First Fundamental Theorem of Asset Pricing, a local martingale measure $\widehat{\mathbb{Q}}$ that is equivalent to $\mathbb{P}^{€}$. Then, the fact that $\mathbb{Q}^{€}|_{\mathcal{F}^0_R} = \widehat{\mathbb{Q}}^{€}|_{\mathcal{F}^0_R}$ can be proven line by line as in Proposition 5. □

This proposition yields directly the minimal replication cost for a claim under the measure $\mathbb{P}$:

Corollary 8 (Minimal replication cost)**.** *Assuming (A), (B), (i), and (ii) in Proposition 6, the minimal replicating cost for a contingent claim under $\mathbb{P}$ is exactly the one computed in Theorem 2.*

Proof. Since $\mathbb{P}$ is equivalent to $\mathbb{P}^{€} + \mathbb{P}^{\$}$, it is equivalent to the sum of the corresponding risk-neutral measures, $\widehat{\mathbb{Q}}^{€} + \mathbb{Q}^{\$}$. By Proposition 6, $\mathbb{P}$ is then equivalent to $\mathbb{Q}^{€} + \mathbb{Q}^{\$}$, which yields the statement. □

We also obtain an interpretation of the strict local martingality of X as the possibility of an explosion under the physical measure:

Corollary 9 (Interpretation of strict local martingality)**.** *Assuming (A), (B), (i), and (ii) in Proposition 6, we have that $\mathbb{P}(X_T = \infty) > 0$ if and only if X is a strict $\mathbb{Q}^{\$}$-local martingale; equivalently, we have that $\mathbb{P}(1/X_T = \infty) > 0$ if and only if $1/X$ is a strict $\mathbb{Q}^{€}$-local martingale.*

Proof. If $\mathbb{P}(X_T = \infty) > 0$, then by definition $\mathbb{P}^{€}$, and thus $\mathbb{Q}^{€}$, are not absolutely continuous with respect to $\mathbb{P}^{\$}$. This implies directly that X cannot be a true $\mathbb{Q}^{\$}$-martingale. The reverse direction follows along the same lines, and so does the second statement of the corollary. □

In Carr et al. (2012) we discuss the so-called "Quadratic Normal Volatility" models to study a whole class of probability measures $\mathbb{P}$ where the assumptions of the last proposition hold. In the following, we discuss examples of physical measures $\mathbb{P}$ that do not satisfy either the existence of an equivalent local martingale measure for $\mathbb{P}^{€}$ or the equivalence of $\mathbb{P}^{\$}$ and $\mathbb{P}^{€}$ before $R \wedge S$:

Example 6 (Reciprocal of a three-dimensional Bessel process (continued)). If we set $\mathbb{P} = \mathbb{Q}^{\$}$ in Example 5, to wit, if X is the reciprocal of a three-dimensional $\mathbb{P}$-Bessel process, then $\mathbb{P}^{€} = \mathbb{P}^{\$} = \mathbb{P}$, which gives (A), (i), and (ii) of Proposition 6. As Delbaen and Schachermayer (1995a) discuss, there is arbitrage with respect to $1/X$ and no equivalent probability measure exists under which $1/X$ is a local martingale. Thus, (B) in Proposition 6 is not satisfied. This observation also follows directly from Corollary 9.

However, if we set $\mathbb{P} = \mathbb{Q}^{€}$ in Example 5, to wit, if X is a $\mathbb{P}$-Brownian motion stopped in zero, then the requirements of Proposition 6 are satisfied. It is clear that (B) and (i) hold. Furthermore, $\mathbb{Q}^{\$}$ defined by $\mathrm{d}\mathbb{Q}^{\$}/\mathrm{d}\mathbb{P} = 1/X_T$ and $\mathbb{P}^{\$}(\cdot) = \mathbb{P}(\cdot|1/X_T > 0)$ are equivalent which yields (A) and (ii). □

Example 7 (Lack of equivalence of $\mathbb{P}^{\$}$ and $\mathbb{P}^{€}$). We fix $n = 2$, $d = 1$, $T = 2$ and choose $\mathbb{P}$ so that $w^{(2)}$ is distributed as at for all $t \in [0, 2]$ where a is either 1 or -1 with probability $1/2$. Furthermore, $X_t = 1$ for $t \in [0, 1]$ and $X_t = Y_{t-1}^{w_1^{(2)}}$, where Y has the same distribution as the process in (9). Thus, $\{X_2 = \infty\} = \{w_1^{(2)} = -1\}$ and $\{X_2 = 0\} = \{w_1^{(2)} = 1\}$. Furthermore, both $\mathbb{P}^{\$}$ and $\mathbb{P}^{€}$ are already the unique martingale measures such that (A), (B), and (ii) of Proposition 6 are satisfied. However, (i) clearly does not hold; indeed the two probability measures $\mathbb{P}^{\$}$ and $\mathbb{P}^{€}$ are singular with respect to each other. □

4.1 Empirical evidence for hyperinflations

Cagan (1956) defines a *hyperinflation* as a price index increase by 50 percent or more within a month. Such an economic event basically corresponds to a complete devaluation of the corresponding domestic numéraire and an explosion of the exchange rate with respect to any other currency provided that the other currency does not also experience a hyperinflation.

In the past century, there have been several examples for such extreme price increases and exchange rate explosions. At the beginning of the the 1920s, hyperinflations happened, among others, in Austria, Germany and Poland. For example, the price of one Dollar, measured in units of the respective domestic currency, went up by a factor of over 4500 in Austria from January 1919 to August 1922 and by a factor of over 10^{10} from January 1922 to December 1923 in Germany; these and many more facts concerning the hyperinflations following World War 1 can be found in Sargent (1982). Hungary experienced one of the most extreme hikes in prices from August 1945 to July 1946. Prices soared by a factor of over 10^{27} in that 12-month period to which the month of July contributed a staggering raise of $4 * 10^{16}$ percent of prices; see Cagan (1987) and Romer (2001). Sachs (1986) discusses another hyperinflation in Bolivia from August 1984 to August 1985. In this period, price levels increased by $20,000$ percent. More recently, price levels of Zimbabwe increased dramatically; for instance, prices there increased by an annualized inflation rate of over $2 * 10^8$ percent in July 2009.[1]

These are only some of the more famous occurrences of hyperinflation in the last century; others have happened, for example, in China, Greece and Argentina; a more complete list can be found on Wikipedia[2].

In this context, Frankel (2005) studies 103 developing countries between 1971 and 2003 and finds 188 currency crashes, which are devaluations of a currency by at least 25 percent within a 12-month period.

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[1] See `http://news.bbc.co.uk/1/hi/world/africa/7660569.stm`, retrieved August 5,2011.

[2] See `http://en.wikipedia.org/wiki/Hyperinflation`, retrieved August 5,2011.

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